Calculus 1, Practise Course

4th week

I. Evaluating limits graphically

1. Sketch the graph of f and use it to make a conjecture about the value of $f(a)$, $\lim_{x\to a} f(x)$, $\lim_{x\to a+} f(x)$, and $\lim_{x\to a-} f(x)$.

   (a)
   $$f(x) = \begin{cases} 3-x, & \text{if } x<2 \\ x-1, & \text{if } x>2 \end{cases}, \quad a=2$$

   (b)
   $$f(x) = \begin{cases} x^2+1, & \text{if } x\le -1 \\ 3, & \text{if } x>-1 \end{cases}, \quad a=-1$$

   (c)
   $$f(x) = \begin{cases} \sqrt{x}, & \text{if } x<4 \\ 3, & \text{if } x=4 \\ x+1, & \text{if } x>4 \end{cases}, \quad a=4$$

   (d) $f(x) = |x+2|+2$, $a=-2$

   (e) $f(x) = \frac{x-100}{\sqrt{x}-10}$, $a=100$

   (f) $f(x) = \frac{|x|}{x}$, $(x\neq 0)$, $a=0$

2. For any real number x, the *greatest integer function* or *floor function*[1] $\lfloor x\rfloor$ is the greatest integer less than or equal to x. Sketch its graph and give the following limits.

   (a) $\lim_{x\to -1-}\lfloor x\rfloor$, $\lim_{x\to -1+}\lfloor x\rfloor$, $\lim_{x\to 2-}\lfloor x\rfloor$, $\lim_{x\to 2+}\lfloor x\rfloor$

[1] greatest integer function = alsó egész függvény, floor =padló

(b) $\lim_{x\to -1.3-}\lfloor x\rfloor$, $\lim_{x\to -1.3+}\lfloor x\rfloor$, $\lim_{x\to 2.7-}\lfloor x\rfloor$, $\lim_{x\to 2.7+}\lfloor x\rfloor$

3. For any real number x, the *smallest integer function* or *ceiling function*[2] $\lceil x\rceil$ is the smallest integer greater than or equal to x. Sketch its graph and give the following limits.

   (a) $\lim_{x\to -1-}\lceil x\rceil$, $\lim_{x\to -1+}\lceil x\rceil$, $\lim_{x\to 2-}\lceil x\rceil$, $\lim_{x\to 2+}\lceil x\rceil$

   (b) $\lim_{x\to -1.3-}\lceil x\rceil$, $\lim_{x\to -1.3+}\lceil x\rceil$, $\lim_{x\to 2.7-}\lceil x\rceil$, $\lim_{x\to 2.7+}\lceil x\rceil$

4. Suppose f is an even function, with $\lim_{x\to 2+} f(x) = 5$ and $\lim_{x\to 2-} f(x) = 8$. Evaluate the following limits.

   (a) $\lim_{x\to -2+} f(x)$

   (b) $\lim_{x\to -2-} f(x)$

5. Suppose f is an odd function, with $\lim_{x\to 2+} f(x) = 5$ and $\lim_{x\to 2-} f(x) = 8$. Evaluate the following limits.

   (a) $\lim_{x\to -2+} f(x)$

   (b) $\lim_{x\to -2-} f(x)$

II. Techniques for computing limits

1. Evaluate the following limits or state that they do not exist.

   (a) $\lim_{x\to 4}\frac{x^2-4x-1}{3x-1}$

   (b) $\lim_{x\to 3}\frac{x^2-7x+12}{x-3}$

   (c) $\lim_{x\to 5}\frac{4x^2-100}{x-5}$

   (d) $\lim_{t\to 3}\sqrt[3]{t^2-10}$

   (e) $\lim_{x\to 4}\frac{x^2-16}{4-x}$

   (f) $\lim_{x\to 3}\frac{x^2-2x-3}{x-3}$

   (g) $\lim_{x\to b}\frac{(x-b)^{50}-x+b}{x-b}$

   (h) $\lim_{x\to -b}\frac{(x+b)^7+(x+b)^{10}}{4(x+b)}$

   (i) $\lim_{x\to -1}\frac{(2x-1)^2-9}{x+1}$

   (j) $\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}$

[2] smallest integer function = felső egész függvény, ceiling = mennyezet

(k) $\lim_{h\to 0} \frac{\frac{1}{5+h}-\frac{1}{5}}{h}$

(l) $\lim_{w\to 1} \left(\frac{1}{w^2-w} - \frac{1}{w-1}\right)$

(m) $\lim_{w\to 5} \left(\frac{1}{w^2-4w-5} - \frac{1}{6(w-5)}\right)$

(n) $\lim_{t\to 3} \left(\left(4t - \frac{2}{t-3}\right)(6-t^2+t)\right)$

(o) $\lim_{x\to a} \frac{x-a}{\sqrt{x}-\sqrt{a}}$, $a > 0$

(p) $\lim_{x\to a} \frac{x^2-a^2}{\sqrt{x}-\sqrt{a}}$, $a > 0$

(q) $\lim_{h\to 0} \frac{\sqrt{16+h}-4}{h}$

(r) $\lim_{x\to 4} \frac{\frac{1}{x}-\frac{1}{4}}{x-4}$

(s) $\lim_{x\to 3} \frac{\frac{1}{x^2+2x}-\frac{1}{15}}{x-3}$

(t) $\lim_{x\to 1} \frac{\sqrt{10x-9}-1}{x-1}$

(u) $\lim_{x\to 2} \left(\frac{1}{x-2} - \frac{2}{x^2-2x}\right)$

(v) $\lim_{x\to 1} \frac{x-1}{\sqrt{4x+5}-3}$

2. Evaluate the following limits with the help of trigonometric identities.

(a) $\lim_{x\to 0} \frac{\sin 2x}{\sin x}$

(b) $\lim_{x\to 0} \frac{1-\cos x}{\cos^2 x-1}$

(c) $\lim_{x\to 0} \frac{1-\cos x}{\cos^2 x-3\cos x+2}$

3. *[3] Calculate the following limits.
Hint: Use the factorization formula

$$x^n - a^n = (x-a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1}).$$

(a) $\lim_{x\to 2} \frac{x^5-32}{x-2}$

(b) $\lim_{x\to 1} \frac{x^6-1}{x-1}$

(c) $\lim_{x\to -1} \frac{x^7+1}{x+1}$

(d) $\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{x-1}$

(e) $\lim_{x\to 16} \frac{\sqrt[4]{x}-2}{x-16}$

(f) $\lim_{x\to a} \frac{x^n-a^n}{x-a}$, n is a positive integer

[3]*: more challanging tasks

4. **[4] Find the following limits.

(a) $\lim_{x\to 1}(\frac{3}{1-x^3} + \frac{1}{x-2})$

(b) $\lim_{x\to 0} \frac{(1+x)^5-1-5x}{x^2+x^5}$

(c) $\lim_{x\to 6} \frac{\sqrt{x-2}-2}{x-6}$

(d) $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1+x^2}}{\sqrt{1+x}-1}$

(e) $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt[3]{1+x}-\sqrt[3]{1-x}}$

(f) $\lim_{x\to 0+} \left(\sqrt{\frac{1}{x} + \sqrt{\frac{1}{x} + \sqrt{\frac{1}{x}}}} - \sqrt{\frac{1}{x}}\right)$

[4] **: problems for brave hearted