Calculus 1, Practise Course

5th week

I. Limits by definitions

1. Proceeding from the definition of the limits prove that

   (a) $\lim_{x\to 1}(3x-8) = -5$

   (b) $\lim_{x\to 2} \frac{3x+1}{5x+4} = \frac{1}{2}$

   (c) $\lim_{x\to 4} \frac{x^2-16}{x^2-4x} = 2$

   (d) Let
   $$f(x) = \begin{cases} 3x-4, & \text{if } x < 0 \\ 2x-4, & \text{if } \geq 0 \end{cases}.$$
   Show that $\lim_{x\to 0+} f(x) = -4$, $\lim_{x\to 0-} f(x) = -4$,$\lim_{x\to 0} f(x) = -4$.

   (e) $\lim_{x\to 1} \frac{1}{(1-x)^2} = \infty$

   (f) $\lim_{x\to \infty} \frac{5x+1}{3x+9} = \frac{5}{3}$

   (g) $\lim_{x\to 1+} \frac{1}{1-x} = -\infty$

   (h) $\lim_{x\to 1-} \frac{1}{1-x} = \infty$

   (i) $\lim_{x\to \infty} \frac{10}{x} = 0$

   (j) $\lim_{x\to \infty} \frac{2x+1}{x} = 2$

   (k) $\lim_{x\to \infty} \frac{x}{1000} = \infty$

   (l) $\lim_{x\to \infty} \frac{x^2+x}{x} = \infty$

2. Suppose that $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = K$. Using the definition of the limit prove that

   (a) $\lim_{x\to x_0} cf(x) = cL$

   (b) $\lim_{x\to x_0} f(x) \cdot g(x) = L \cdot K$

(c) $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{L}{K}$ provided that $K \neq 0$

(d) $\lim_{x\to x_0} \sqrt[n]{f(x)} = \sqrt[n]{L}$ provided that $f(x) > 0$ near x_0.

II. Some trigonometric limits

1. Using the facts that $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and $\lim_{x\to 0} \cos x = 1$ find the following limits.

   (a) $\lim_{x\to 0} \frac{\sin 4x}{x}$

   (b) $\lim_{x\to 0} \frac{\sin 4x}{\sin 7x}$

   (c) $\lim_{x\to 0} \frac{\sin ax}{\sin bx}$

   (d) $\lim_{x\to 0} \frac{\sin^2 2x}{x}$

   (e) $\lim_{x\to 0} \frac{\sin^2 2x}{x^2}$

   (f) * $\lim_{x\to 0} \frac{\tan x - \sin x}{x^3}$

   (g) *$\lim_{x\to 0} \frac{\sqrt{1+\tan x}}{\sin x}$

   (h) *$\lim_{x\to 0}(\frac{1}{\sin x} - \frac{1}{\tan x})$

   (i) *$\lim_{x\to 0} \frac{1-\cos x}{x^2}$

   (j) *$\lim_{x\to 1} \frac{\sin 7\pi x}{\sin 3\pi x}$

   (k) **$\lim_{x\to 1} \frac{\cos(\pi x/2)}{1-x}$
   Hint: Use the substitution $z = 1 - x$.

   (l) **$\lim_{x\to 0} \frac{x \sin x}{1-\cos x}$

   (m) **$\lim_{x\to 0} \frac{\tan^2 x + 2x}{x + x^2}$

   (n) **$\lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$

   (o) *$\lim_{x\to 1} \frac{\sin(x^2-1)}{x-1}$

III. Infinite limits

1. Determine the following limits.

   (a) $\lim_{x\to 2+} \frac{1}{x-2}$, $\lim_{x\to 2-} \frac{1}{x-2}$, $\lim_{x\to 2} \frac{1}{x-2}$

   (b) $\lim_{x\to 3+} \frac{2x}{(x-3)^2}$, $\lim_{x\to 3-} \frac{2x}{(x-3)^2}$, $\lim_{x\to 3} \frac{2x}{(x-3)^2}$

   (c) $\lim_{x\to 1+} \frac{x}{|x-1|}$, $\lim_{x\to 1-} \frac{x}{|x-1|}$, $\lim_{x\to 1} \frac{x}{|x-1|}$

   (d) $\lim_{t\to 3+} \frac{1}{\sqrt{t(t-3)}}$, $\lim_{t\to 3-} \frac{1}{\sqrt{t(t-3)}}$, $\lim_{t\to 3} \frac{1}{\sqrt{t(t-3)}}$

(e) $\lim_{x\to -2+} \frac{x-4}{x(x+2)}$, $\lim_{x\to -2-} \frac{x-4}{x(x+2)}$, $\lim_{x\to -2} \frac{x-4}{x(x+2)}$

(f) $\lim_{x\to 1+} \frac{x-3}{\sqrt{x^2-5x+4}}$, $\lim_{x\to 1-} \frac{x-3}{\sqrt{x^2-5x+4}}$, $\lim_{x\to 1} \frac{x-3}{\sqrt{x^2-5x+4}}$

(g) $\lim_{x\to 0} \frac{x-3}{x^4-9x^2}$

(h) $\lim_{x\to 3} \frac{x-3}{x^4-9x^2}$

(i) $\lim_{x\to -3} \frac{x-3}{x^4-9x^2}$

(j) *$\lim_{x\to 1+} 3^{\frac{1}{x-1}}$, $\lim_{x\to 1-} 3^{\frac{1}{x-1}}$

(k) *$\lim_{x\to -1+} 3^{\frac{1}{x+1}}$, $\lim_{x\to -1-} 3^{\frac{1}{x+1}}$

(l) **$\lim_{x\to 1+} 3^{\frac{x-1}{(1-x)^2}}$, $\lim_{x\to 1-} 3^{\frac{x-1}{(1-x)^2}}$