Calculus 1, Practise Course

7th week

I. Intermediate value theorem

1. Show that the following equations have a solution on the given interval.
   (a) $2x^3 + x - 2 = 0$ on $(-1, 1)$
   (b) $\sqrt{x^4 + 25x^3 + 10} = 5$ on $(0, 1)$
   (c) $x^3 - 5x^2 + 2x = -1$ on $(-1, 5)$
   (d) $-x^5 - 4x^2 + 2\sqrt{x} + 5 = 0$ on $(0, 3)$
   (e) $x + e^x = 0$ on $(-1, 0)$
2. For each of the following polynomials p, find an integer n such that $p(x) = 0$ for some x between n and $n + 1$.
   (a) $p(x) = x^3 - x + 3$
   (b) $p(x) = x^5 + x + 1$
   (c) $p(x) = x^5 + 5x^4 + 2x + 1$
3. * Show that the equation
   $$x^3 - 3x + 1 = 0$$
   has one root on the interval $[1, 2]$. Calculate this root approximately to within two decimal places. (The more diligent can write a program for the problem.)
4. ** Prove that there is some number x such that
   $$x^{179} + \frac{163}{1 + x^2 + \sin^2 x} = 119.$$

II. Using the definition of derivative

1. Use the definition of the derivative to determine $f'(x)$ if
   (a) $f(x) = ax^2 + bx + c$, where a, b, c are constants
   (b) $f(x) = \sqrt{ax+b}$, where a, b are constants
2. Give the values of parameters a and b to make f differentiable at x_0

$$f(x) = \begin{cases} x^2, & \text{ha } x \leq x_0 \\ ax+b, & \text{ha } x > x_0 \end{cases}$$

3. Give the values of parameters a, b and c to make f differentiable at 0

$$f(x) = \begin{cases} e^{2x}, & \text{ha } x \geq 0 \\ ax^2+bx+c, & \text{ha } x < 0 \end{cases}$$

4. * Suppose that f is differentiable everywhere. Prove that
   (a) if f is even, then f' is odd.
   (b) if f is odd, then f' is even.

   ** What can we tell about the parity of $f^{(k)}$ (the kth derivative of f)?
5. ** Suppose that f is differentiable at x. Prove that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}.$$

III. Rules of differentiation

1. Find the derivative of the following functions.
   (a) $f(x) = x^{100} + 6e^x + \sqrt[3]{x}$
   (b) $f(x) = \frac{6}{\sqrt{x}} + \frac{9}{\sqrt[5]{x^3}} - x^{7/3}$
   (c) $s(t) = 4\sqrt{t} - 9\sin t + 7 \cdot 4^t + e^4$
   (d) $f(x) = (\sqrt{x}+1)(3x^2+2)$
   (e) $g(t) = \sqrt{t}(\sqrt[3]{t} - t^{5/2})$
   (f) $f(x) = e^x \sin x$

(g) $f(x) = 2^x(x^7 - e^x)$

(h) $f(x) = \frac{x^3-6x^2+8x}{x^2-2x}$

(i) $y(x) = \frac{x+\sqrt{x}+\ln x}{x-2\sqrt[3]{x}}$

(j) $z(t) = \frac{e^{2t}+3e^t+2}{e^{-t}+2}$

(k)

$$f(x) = \begin{cases} x^2+1, & \text{if } x \leq 0 \\ 2x^2+x+1, & \text{if } x > 0 \end{cases}, \quad x_0 = 1.$$

(l)

$$f(x) = \begin{cases} x+5e^x, & \text{if } x \leq 1 \\ 2x^3+4x+5, & \text{if } x > 1 \end{cases}, \quad x_0 = 3.$$

(m) $f(x) = (3x^2 + 7x)^{10}$

(n) $g(x) = \sin^5 2x$

(o) $g(x) = \sqrt[3]{x^2 - 2x - 3}$

(p) $f(x) = \sqrt{x + \sqrt{x}}$

(q) $f(x) = \cos(x^6 + x^2 e^{-x})$

(r) $h(t) = \tan\left(\frac{x^5+7\ln x}{x+e^{-2x}}\right)$

(s) $h(x) = e^{\sqrt{x-1}}$

(t) $f(x) = \sin\left(\frac{\cos 2x}{x}\right)$

(u) $k(x) = \sin(\cos(\sin x))$

(v) $f(t) = e^{\frac{1}{\sqrt{t-1}}}$

(w) * $f(x) = \sin(x^2 + \sin(x^2 + \sin x^2))$

(x) * $f(x) = (((x^2 + x)^3 + x)^4 + x)^5$

(y) ** $f(x) = \sin\left(\frac{x^3}{\sin\left(\frac{x^3}{\sin x}\right)}\right)$

(z) ** $f(x) = \sin\left(\frac{x}{x-\sin\left(\frac{x}{x-\sin x}\right)}\right)$