Calculus 1, Practise Course

8th week

I. Derivatives

1. Investigate the following functions for differentiability. Give the derivatives too.
   (a) $f(x) = \sqrt[3]{(x+2)^2}$
   (b) $f(x) = \sqrt{1-x^2}$
   (c) $f(x) = \sqrt{\frac{x+1}{x-1}}$
   (d) * $f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{ha } x \neq 0 \\ 0, & \text{ha } x = 0 \end{cases}$
   (e) $f(x) = (\frac{ax+b}{cx+d})^{-2/3}, \quad a, b, c, d \in \mathbb{R}$
   (f) $f(x) = \arcsin(\cos x)$

2. Show that the function $y = xe^{-x^2/2}$ satisfies the equation $xy' = (1-x^2)y$.

3. Show that the function $y = xe^{-x}$ satisfies the equation

$$xy' = (1-x)y.$$

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function, $g(x) = \sin^2 x$ and $h(x) = \cos^2 x$. Give the derivative of the function $F = (f \circ g) + (g \circ h)$.

5. Give a closed expression for the following sums with the help of the derivatives.
   (a) $F(x) = 1 + 2x + \cdots + nx^{n-1}, \quad n \geq 2$
   (b) $G(x) = 1 + 2^2x + \ldots n^2x^{n-1}, \quad n \geq 2$
   (*Hint:* Observe that $G(x) = (xF(x))'$)

6. * Find a formula for the nth derivative of $y = \frac{1}{x(1-x)}$.

7. * Consider the function

$$f(x) = \begin{cases} x^2, & \text{if, } x \geq 0, \\ -x^2, & \text{if } x < 0. \end{cases}$$

Show that $f''(0)$ does not exist.

8. * Find all derivatives of $y = 2x^2 + x - 1 + \frac{1}{x}$.

II. Tangent lines

1. Determine the tangent lines of the following functions corresponding to the given x_0 value.

   (a) $f(x) = \sin\sqrt{x}, \quad x_0 = \pi^2$

   (b) $f(x) = x^3 - 8x, \quad x_0 = 3$

   (c) $f(x) = e^{\sin x}, \quad x_0 = \pi.$

   (d) $f(x) = \frac{(x^2-1)^2}{x^3-6x-1}$, $x_0 = 0$

2. What can we say about a, b and c, if the $f(x) = ax^2 + bx + c$ parabola touches the x axis, i.e. one of its tangent line is the x axis?

3. * Prove that if the graph of $f(x) = x^3 + px + q$ touches the x axis, then $(p/3)^3 + (q/2)^2 = 0$.

4. ** Give the values of parameters a and b to make f continous everywhere and having tangent line everywhere.

$$f(x) = \begin{cases} \frac{m^2}{|x|}, & \text{ha } |x| > c \\ ax^2 + b, & \text{ha } |x| \leq c \end{cases}$$

5. A straight line passing through the point of contact perpendicularly to the tangent line is called the **normal to the curve**. The equation of the normal of the function $y = y(x)$ at the point $P(x_0, y_0)$ is given by

$$y = y_0 - \frac{1}{y'(x_0)}(x - x_0), \quad y'(x_0) \neq 0.$$

   Write the equations of the tangent line an the normal:

   (a) to the curve $y = x^3 - 3x + 2$ at the point $P(2, 4)$

   (b) to the parabola $y = 2x^2 - x + 5$ at $x = -\frac{1}{2}$.

(c) * to the curve $y = x^4 + 3x^2 - 16$ at the points of intersection with the parabola $y = 3x^2$.

III. Logarithmic differentiation

1. Find the derivative of the following functions.

   (a) $f(x) = x^{10x}$, $(x > 0)$

   (b) $f(x) = (2x)^{2x}$, $(x > 0)$

   (c) $f(x) = (\sin x)^{\tan x}$

   (d) * $f(x) = x^x + x^{x^x}, \quad x > 0$

   (e) $f(x) = \left(\frac{x^8(x+7)^{1/3}\cos^3 x}{\sqrt{x-1}}\right)^6$

   (f) $f(x) = \sqrt[3]{\frac{x^2(x+1)}{(x-2)(x^2+2)(x+3)}}$

2. Find an equation of the tangent line to $y = x^{\sqrt{x}}$ at $x = 4$. Determine whether the graph of the function has any horizontal tangent line.

3. Find an equation of the line tangent to $y = x^{\sin x}$ at the point $x = 1$.

IV. Implicit differentiation

1. Use implicit differentiation to find $\frac{\mathrm{d}\,y}{\mathrm{d}\,x}$.

   (a) $\sin x + \sin y = y$

   (b) $6x^3 + 7y^3 = 13xy$

   (c) $e^{xy} = 2y$

   (d) $\sin xy = x + y$

   (e) $\sqrt{x + y^2} = \sin y$

   (f) $\ln x + e^{-y/x} = 5$

   (g) $e^x \sin y - e^{-y} \cos x = 0$

2. Determine an equation of the tangent line to the curve at the given point.

   (a) $\sin y + 5x = y^2$, $(0, 0)$

   (b) $x^3 + y^3 = 2xy$, $(1, 1)$

   (c) $x^2 + xy + y^2 = 7$, $(2, 1)$

(d) $(x^2 + y^2)^2 = \frac{25xy^2}{4}$, $(1, 2)$

(e) $\cos(x - y) + \sin y = \sqrt{2}$, $(\pi/2, \pi/4)$

(f) $x + y^3 - y = 1$, $x = 1$

(g) $4x^3 = y^2(4 - x)$, $x = 2$

3. Find $\frac{\mathrm{d}^2 y}{\mathrm{d} x^2}$.

   (a) $x + y^2 = 1$

   (b) $e^{2y} + x = y$

   (c) $x + y = e^{x-y}$

   (d) $x^3 - y^3 = 1$

4. ** Show that the ellipse $4x^2 + 9y^2 = 45$ and the hyperbola $x^2 - 4y^2 = 5$ are orthogonal (perpendicular).

5. ** Show that the parabolas $y^2 = 4x + 4$ and $y^2 = 4 - 4x$ intersect at right angles.

6. ** Show that the circles $x^2 + y^2 - 12x - 6y + 25 = 0$ and $x^2 + y^2 + 2x + y - 10 = 0$ are tangent to each other at the point $(2, 1)$.