Calculus 1, Practise Course

9th week

I. Using the derivatives

1. The following limits represents $f'(a)$ for some function f and some real numbers a. Evaluate the limits with the help of this knowledge.
   (a) $\lim_{x\to 0}\frac{x+e^x-1}{x}$
   (b) $\lim_{h\to 0}\frac{\sqrt[3]{27+h}-3}{h}$
   (c) $\lim_{x\to 2}\frac{(x^4-15)^7-1}{x-2}$
   (d) $\lim_{x\to \pi}\frac{x\cos x+\pi}{x-\pi}$
2. Suppose $f'(x) < 2$, for all $x \geq 2$, and $f(2) = 7$. Show that $f(4) < 11$.
3. Suppose $f'(x) > 1$, for all $x > 0$, and $f(0) = 0$. Show that $f(x) < x$, for all $x > 0$.
4. * We know that $(e^x)' = e^x$ for all x. Are there any more functions tha coincide with their derivatives everywhere?
   (*Hint:* Supposing that $f'(x) = f(x)$ for all x, intvestigate the function $g(x) = \frac{f(x)}{e^x}$.)

II. Derivatives of inverse functions, derivatives of arcus functions

1. Consider the following functions. Without finding the inverse, evaluate the derivative of the inverse at the given point.
   (a) $f(x) = \ln(5x + e)$ $(1, 0)$
   (b) $f(x) = x^2 - 2x - 3$, for $x \leq 1$, $(12, -3)$
   (c) $f(x) = (x + 2)^2$, $(36, 4)$
   (d) $f(x) = \log_{10} 3x$, $(0, 1/3)$
2. Evaluate the derivative of the following functions.

(a) $f(x) = \arcsin 4x$

(b) $f(x) = \arccos(e^{-2x})$

(c) $f(x) = \arcsin(e^{\sin x})$

(d) $g(t) = 2t \operatorname{arctg} t - \ln(1 + t^2)$

(e) $f(z) = \ln(\operatorname{arctg} z)$

(f) $f(x) = \sin(\operatorname{arctg}(\ln x))$

(g) $f(x) = \frac{1}{\operatorname{arctg}(x^2+4)}$

(h) $f(x) = \arccos(1/x)$

3. * Prove the following formulas with the help of derivatives.

   (a) $\arcsin x + \arccos x = \pi/2$
   (*Hint:* Derivate the function $f(x) = \arcsin x + \arccos x = \pi/2$)

   (b) $2\sin^2 x = 1 - \cos 2x$

   (c) $\arccos x \frac{1-x^2}{1+x^2} = 2 \operatorname{arctg} x$, $x \geq 0$

4. * Use derivatives to show that $\operatorname{arctg} \frac{2}{x^2}$ and $\operatorname{arctg}(x + 1) - \operatorname{arctg}(x - 1)$ differ by a constant. Moreover

$$\operatorname{arctg} \frac{2}{x^2} = \operatorname{arctg}(x + 1) - \operatorname{arctg}(x - 1), \quad \text{for } x \neq 0.$$

III. Mean value theorems (Rolle, Lagrange, Cauchy)

1. Prove that the polynomial $f(x) = x^7 + 14x - 3$ has got exactly one root.

2. Consider the polynomial $p(x) = 5x^3 - 2x^2 + 3x - 4$. Prove that $p(x)$ has a zero between 0 and 1 that is the only zero of $p(x)$.

3. Show that the equation $3 \tan x + x^3 = 2$ has exactly one solution on the interval $[0, \pi/4]$.

4. Prove that $x^3 + px + q = 0$ has exactly one real root if $p > 0$.

5. Prove that the polynomial $f(x) = x^n + ax + b$ has got at least two real roots if n is even and at least three, if n is odd.

6. Prove that $|\sin x - \sin y| \leq |x - y|$.

7. Prove that $|\tan x + \tan y| \geq |x + y|$, if $x, y \in (-\pi/2, \pi/2)$.

8. Prove that $\frac{x}{1+x} < \ln(1+x) < x$, if $x > 0$. Give an estimation for the value $\ln(1+x)$.

9. ** Let f be an infinitely differentiable function. Suppose that, for some positive integer n,
$$f(1) = f(0) = f'(0) = f''(0) = \cdots = f^{(n)}(0) = 0.$$
Prove that $f^{(n+1)}(x) = 0$ for some x on $(0, 1)$.

10. Find an equation of the tangent line to $y = x^{\sqrt{x}}$ at $x = 4$. Determine whether the graph of the function has any horizontal tangent line.

11. Find an equation of the line tangent to $y = x^{\sin x}$ at the point $x = 1$.

IV. Maxima and minima

1. Find the critical points of the following functions.
   (a) $f(x) = 3x^3 + \frac{3x^2}{2} - 2x$
   (b) $f(x) = \frac{x^4}{4} - \frac{x^3}{3} - 3x^2 + 10$
   (c) $f(x) = x^2\sqrt{x+5}$
   (d) $g(x) = \frac{x}{\sqrt{x-10}}$
   (e) $f(x) = (\arcsin x)(\arccos x)$
   (f) $f(t) = t^2 - 2\ln(t^2+1)$
   (g) $f(x) = x - 5 \operatorname{arctg} x$

2. Determine the location and value of the absolute extreme values of f on the given interval, if they exist.
   (a) $f(x) = x^3 - 3x^2$ on $[-1, 3]$
   (b) $f(x) = 2x^6 - 15x^4 + 24x^2$ on $[-2, 2]$
   (c) $f(x) = \frac{x}{(x^2+3)^2}$ on $[-2, 2]$
   (d) $f(x) = (2x)^x$ on $[0.1, 1]$
   (e) $f(x) = x^2 + \arccos x$ on $[-1, 1]$
   (f) $f(x) = x^3 e^{-x}$ on $[-1, 5]$
   (g) $f(t) = \frac{3t}{t^2+1}$ on $[-2, 2]$
   (h) $f(x) = x - \sin x$ on $[0, \pi/2]$
   (i) $f(x) = \begin{cases} x^3 - \frac{x}{3} & \text{if } 0 \le x \le 1 \\ x^2 + x - \frac{4}{3} & \text{if } 1 < x \le 2 \end{cases}$ on $[0, 2]$

3. * Prove that for all real x

$$\frac{2}{3} \leq \frac{x^2+1}{x^2+x+1} \leq 2.$$

V. Increasing and decreasing functions, convexity, concavity

1. Find the intervals on which f is increasing and the intervals on which it is decrasing. Locate the local minimum and maximum values.

   (a) $f(x) = x^3 + 4x$

   (b) $f(x) = -\frac{x^3}{3} + \frac{x^2}{2} + 2x$

   (c) $f(x) = 2x^5 - \frac{15x^4}{4} + \frac{5x^3}{3}$

   (d) $f(x) = \frac{e^x}{e^{2x}+1}$

   (e) $f(x) = x^2 \ln x^2 + 1$

   (f) $f(x) = x^{2/3}(x^2 - 4)$

   (g) $f(x) = -2\cos x - x$ on $[0, 2\pi]$

   (h) $f(x) = \sqrt{9-x^2} + \arcsin(x/3)$

   (i) $f(x) = \frac{x^2}{x^2-1}$

   (j) $f(x) = \arctan\left(\frac{x}{x^2+2}\right)$

2. Determine the intervals on which the following functions are convex or concave.

   (a) $f(x) = x^4 - 2x^3 + 1$

   (b) $f(x) = 5x^4 - 20x^3 + 10$

   (c) $f(x) = \frac{1}{1+x^2}$

   (d) $f(x) = e^x(x-3)$

   (e) $f(x) = \sqrt{x}\ln x$

   (f) $f(x) = \sqrt[3]{x-4}$

   (g) $f(x) = x^4 e^x + x$

   (h) $f(x) = 2x^2 \ln x - 5x^2$

   (i) $f(t) = 2 + \cos 2t$ on $[0, \pi]$