Calculus 1, Practise Course

10th week

I. l'Hospital's Rule

1. Evaluate the following limits. Use l'Hospital's rule when it is convenient and applicable.

   (a) $\lim_{x\to 1} \dfrac{\ln x}{x-1}$,

   (b) $\lim_{x\to 0} \dfrac{e^{ax}-e^{-ax}}{\ln(1+x)}$

   (c) $\lim_{x\to 0} \dfrac{e^{x}-e^{-x}-2x}{x-\sin x}$

   (d) $\lim_{x\to \infty} \dfrac{xe^{x/2}}{e^{x}+x}$

   (e) $\lim_{x\to \infty} \dfrac{\ln x^2}{\sqrt{x}}$

   (f) $\lim_{x\to 0+} x\ln\sin x$

   (g) $\lim_{x\to 0+} \dfrac{\ln x}{1+\ln\sin x}$

   (h) $\lim_{x\to 0}(\arcsin x)(\cot x)$

   (i) $\lim_{x\to -\infty} x^2e^x$

   (j) $\lim_{x\to 0}(1/x-1/(e^x-1))$

   (k) $\lim_{x\to 1}(1/\ln x-1/(x-1))$

   (l) $\lim_{x\to 0} \left(\dfrac{1}{x}\right)^{\sin x}$

   (m) $\lim_{x\to 0}(\sin x)^x$

   (n) $\lim_{x\to 0+}(1+x)^{\ln x}$

(o) $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x$, where a is a constant

(p) $\lim_{x\to 0}(e^{ax}+x)^{1/x}$, where a is a constant

(q) $\lim_{x\to\infty}(\log_2 x - \log_3 x)$

(r) *$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2}$

(s) *$\lim_{x\to 0+} x^{1/(1+\ln x)}$

(t) *$\lim_{x\to 0}(1+ax)^{b/x}$

2. Use limit methods to determine which of the two given functions grows faster, or state that they have comparable growth rates.

(a) $x^2\ln x$, $\ln^2 x$

(b) $x^2\ln x$, x^3

(c) x^{20}, 1.0000001^x

(d) e^{x^2}, $x^{x/10}$

(e) $\ln x$, $\ln(\ln x)$

II. Graphing functions

1. Sketch the graph of the following functions.

(a) $f(x)=x^3-6x^2+9x$

(b) $f(x)=3\sqrt{x}-x^{3/2}$

(c) $f(x)=\frac{x^2+12}{2x+1}$

(d) $f(x)=x^6-3x^4+3x^2-1$

(e) $f(x)=x-2\arctan\frac{x}{x+1}$

(f) $f(x)=x^2e^{1/x}$

(g) $f(x)=x\sqrt{16-x^2}$

(h) $f(x)=x^2\sqrt{1-x}$

(i) $f(x)=x^2\ln x^2$

(j) $f(x)=\frac{x}{1+x^2}$

(k) $f(x)=\ln(x^2+1)$

(l) $f(x)=\sin x - x$ on $[0,2\pi]$

(m) $f(x)=x\sqrt{x+3}$

(n) $f(x) = \frac{1}{e^{-x}-1}$

III. Optimization problems

1. Of all rectangles with a fixed perimeter P, which one has the maximum area?
2. Of all rectangles with a fixed area A, which one has the minimum perimeter?
3. Find positive numbers x and y satisfying the equation $xy = 12$ such that the sum $2x + y$ is a small as possible.
4. Find the point(s) on the hyperbola $x^2 - y^2 = 2$ closest to the point $(0, 1)$
5. A closed box with a square base contain 252 m^3. The bottom costs 5 thousand HUF per m^2, the top costs 2 thousand HUF per m^2, and the sides cost 3 thousand HUF per m^2. Find the dimensions that will minimize the cost.
6. Find the dimensions of the closed cylindrical can that will have a capacity of k units of volume and will use the minimum amount of material. Find the ratio of the height h to the radius r of the top and the bottom.
7. The selling price P of an item is $100 - 0.12x$ dollars, where x is the number of iems produced per day. If the cost C of producing and selling x items is $40x + 15000$ dollars per day, how many items should be produced and sold every day in order to maximize the profit?
8. * Find the point(s) on the graph of $3x^2 + 10xy + 3y^2 = 9$ closest to the origin.
9. * Find the height h and radius r of a cylinder of greatest volume that can be cut within a sphere of radius R.
10. The sum of the squares of two nonnegative numbers is to be 4. How should they be chosen so that the product of their cube is a maximum?