

Calculus 1, Practise Course

12th week

I. Integrals with substitutions

1. Evaluate the following integrals. Apply the fact that if $F'(x) = f(x)$ then $\int f(g(x)) \cdot g'(x) \, dx = F(g(x)) + C$.

(a) $\int \frac{e^{\tan x}}{\cos^2 x} \, dx$

(b) $\int \frac{e^x}{\sqrt{1-e^{2x}}} \, dx$

(c) $\int x \sin x^2 \, dx$

(d) $\int \frac{\operatorname{sh} \sqrt{x}}{\sqrt{x}} \, dx$

(e) $\int \frac{\ln \sqrt{x}}{x} \, dx$

(f) $\int x^2 4^{x^3} \, dx$

(g) $\int 6xe^{-x^2} \, dx$

2. Evaluate the following integrals with the proposed substitutions.

(a) $\int x\sqrt{x-5} \, dx, t = \sqrt{x-5}$

(b) $\int \frac{dx}{1+e^x}, t = 1 + e^x$

(c) $\int \frac{1+\ln x}{3+x \ln x} \, dx, u = 3 + x \ln x$

(d) $\int \frac{\cos \sqrt{x}}{\sqrt{x}} \, dx, u = \sqrt{x}$

(e) $\int \frac{x}{\sqrt{x+1}} \, dx, t = x + 1$

(f) $\int x^2 \sqrt{1-x} \, dx, t = 1 - x$

(g) $\int \csc^6 x \cot x \, dx, u = \csc x$

(h) $\int \frac{1}{\sqrt{x(1-\sqrt{x})}} \, dx, t = 1 - \sqrt{x}$

(i) $\int \frac{e^{2x}}{e^x+1} \, dx, t = e^x + 1$

3. *If necessary, use two or more substitutions to find the following integrals.

(a) $\int x \sin^4 x^2 \cos x^2 \, dx$
(*Hint:* Begin with $u = x^2$, and then use $t = \sin u$.)

(b) $\int \frac{1}{\sqrt{1+\sqrt{1+x}}} \, dx$
(*Hint:* Begin with $u = \sqrt{1+x}$.)

(c) $\int \tan^{10} 4x \sec^2 4x \, dx$
(*Hint:* Begin with $u = 4x$.)

II. Integration of trigonometric functions

1. Evaluate the following integrals with the help of the following identities.

$$\sin ax \sin bx = \frac{1}{2}(\cos(a-b)x - \cos(a+b)x)$$

$$\cos ax \cos bx = \frac{1}{2}(\cos(a-b)x + \cos(a+b)x)$$

$$\sin ax \cos bx = \frac{1}{2}(\sin(a-b)x + \sin(a+b)x)$$

(a) $\int \sin 9x \sin 2x \, dx$

(b) $\int \sin 5x \cos 3x \, dx$

(c) $\int \cos \pi x \sin 6\pi x \, dx$

(d) $\int \cos 7x \cos 9x \, dx$

(e) $\int \cos 11x \sin(-5x) \, dx$

2. Consider the integral $\int \sin^m x \cos^n x \, dx$, where m and n are integers. We showed that the following substitutions can solve the problems:

- If m is odd positive, then apply the substitution $t = \cos x$.
- If n is odd positive, then apply the substitution $t = \sin x$.
- If $m+n$ is an even negative, then apply the substitution $t = \tan x$ or $t = \cot x$.
- If m and n are even non-negative, then use the formulas

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}.$$

Evaluate the following integrals.

- (a) $\int \sin^4 x \cos^3 x \, dx$
- (b) $\int \cos^4 x \sin^2 x \, dx$
- (c) $\int \cos^4 x \, dx$
- (d) $\int \sin^6 x \, dx$
- (e) $\int \sin^5 x \cos^4 x \, dx$
- (f) $\int \sin^6 x \, dx$
- (g) $\int \frac{\sin^2 x}{\cos^6 x} \, dx$
- (h) $\int \frac{\cos^3 x}{\sin^6 x} \, dx$
- (i) $\int \frac{dx}{\cos^4 x}$
- (j) $\int \cos^5 x \, dx$
- (k) $\int \sin^3 x \, dx$

III. Substitutions with trigonometric functions

1. Recall that if the integral contains

- $a^2 - x^2$, then the corresponding substitution is $x = a \sin t$
- $a^2 + x^2$, then the corresponding substitution is $x = a \tan t$
- $x^2 - a^2$, then the corresponding substitution is $x = a \sec t = \frac{a}{\cos t}$.

Evaluate the following integrals. Drawing the corresponding right triangle is very useful!

- (a) $\int \frac{1}{\sqrt{25-x^2}} \, dx$
- (b) $\int \frac{\sqrt{1-x^2}}{x^2} \, dx$
- (c) $\int \frac{x^2}{\sqrt{1-x^2}} \, dx$
- (d) $\int \frac{1}{(1+x^2)^{3/2}} \, dx$
- (e) $\int \frac{1}{x^2\sqrt{x^2+9}} \, dx$
- (f) $\int \frac{x^2}{(25+x^2)^2} \, dx$
- (g) $\int \frac{\sqrt{x^2-1}}{x} \, dx$
- (h) $\int \frac{1}{\sqrt{x^2-64}} \, dx$
- (i) $\int \frac{\sqrt{x^2-9}}{x} \, dx$

IV. Integration of rational functions

1. Find the following integrals

(a) $\int \frac{4x-2}{x^3-x} dx$

(b) $\int \frac{1}{x^4-10x^2+9} dx$

(c) $\int \frac{81}{x^3-9x^2} dx$

(d) $\int \frac{x-5}{x^2(x+1)} dx$

(e) $\int \frac{x-10x^2+27x}{x^2-10x+25} dx$

(f) $\int \frac{20x}{(x-1)(x^2+4x+5)} dx$

(g) $\int \frac{1}{x^2-2x+3} dx$

(h) $\int \frac{1}{x^4-x^2} dx$

(i) $\int \frac{2x+3}{x^2+3x-10} dx$

(j) $\int \frac{x^3-2x^2+4}{x^3(x-2)^2} dx$

(k) $\int \frac{x^4+1}{x^3+9x} dx$

(l) $\int \frac{2}{x(x^2+1)^2} dx$

2. Find the following integrals with the appropriate substitution.

(a) $\int \frac{\sqrt{x}}{x+1} dx$

(b) $\int \frac{6 dx}{e^x-3}$

(c) $\int \frac{4 dx}{e^{2x}-4}$

(d) $\int \frac{1+\sin x}{1-\cos x}$

(e) $\int x\sqrt{5x+3} dx$

(f) $\int \frac{e^{3x}+e^{2x}+e^x}{(e^{2x}+1)^2} dx$

(g) $\int \frac{dx}{x\sqrt{1+2x}}$

(h) $\int \frac{e^x}{(e^x+1)(e^x+2)} dx$