Functional Analysis,
Exercises 1.

1. Let (X, d) be a metric space and (x_n),(y_n) be sequences in X s.t. $x_n \to^d x$ and $y_n \to^d y$. Show that $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$.

2. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing, then

$$d_f(x, y) = |f(x) - f(y)|$$

   defines a metic on $\mathbb{R}$.

3. Consider the sequence space

$$X = \{(x_n)_{n\in\mathbb{N}} : \exists N \in \mathbb{N} \text{ s.t. } x_k = 0 \text{ if } k \geq N\}.$$

   Prove that the metric space (X, d_∞) is not complete.

4. Prove that the set

$$H = \{f \in C([0, 1]) : 0 = f(0) \leq f(x) \leq f(1), x \in [0, 1]\}$$

   is not compact in the metric space $(C([0, 1], d_\infty)$.

5. Let

$$\mathcal{P} = \{f : \mathbb{R} \to \mathbb{R} : \; f \text{ is polinomial}\}$$

   be the vector space of polinomials.

   (a) Prove that for any $f \in \mathcal{P}$, $f(x) = \sum_{k=0}^{n} a_k x^k$ the expression

$$\|f\| := \max\{|a_k| \in \mathbb{R} : k = 1, \ldots, n\}$$

   defines a norm on $\mathcal{P}$.

   (b) Prove that for any $f \in \mathcal{P}$, $\|f\| := \sum_{n=1}^{\infty} \frac{|f(n)|}{n!}$ defines a norm on $\mathcal{P}$.

6. For two vectors $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$ we define a metric on $\mathbb{R}^2$ by

$$d(x,y) = \begin{cases} 1, & \text{if } x_1 \neq x_2, \\ \min\{1, |x_2 - y_2|\}, & \text{if } x_1 = y_1. \end{cases}$$

   Show that d is not induced by any norm.

7. Show that the set

$$A = \{x = (x_n) \in \ell_2 : |x_n| \leq \frac{1}{\sqrt{n}}, n \in \mathbb{N}\}$$

   is not compact in ℓ_2.
   (Hint: show that A is not bounded.)

8. Let $p \in (1, \infty)$ and let q be its conjugate index, i.e. $1/p + 1/q = 1$. Show that if $x = (x_n) \in \ell_p$, then $(|x_n|^{p-1}) \in \ell_q$.

9. Recall the following notations for sequence spaces. If $x = (x_n)_{n \in \mathbb{N}}$ is a sequence, where $x_n \in \mathbb{K}$ $(n \in \mathbb{N})$, then

$$\begin{aligned} \ell_\infty &= \{x : \|x\|_\infty = \sup_j |x_j| < \infty\}, \\ c &= \{x : x \text{ is convergent}\}, \\ c_0 &= \{x : \lim_n x_n = 0\}, \\ c_{00} &= \{x : x_n = 0, \text{ if } n > N \text{ for some } N\}. \end{aligned}$$

   Prove the following propositions:

   (a) c is a closed subspace in ℓ_∞.
   (b) c_0 is a closed subspace in ℓ_∞.
   (c) c_{00} is a linear subspace in ℓ_∞, but is not closed.
   (d) $\ell_p \subset c_0 \subset \ell_\infty$ and the inclusion is proper, $(p \in [1, \infty))$
   (e) ℓ_p is not closed in ℓ_∞, if $p \in [1, \infty)$.
   (f) If $1 \leq p < p' \leq \infty$, then $\ell_p \subset \ell_{p'}$.
   (g) $\lim_{p \to \infty} \|x\| = \|x\|_\infty$ holds for any $x \in \ell_1$.

10. Show that if $0 < p < q < \infty$ then ℓ_p is dense in ℓ_q.

11. Prove that on

$$X := \{f \in C^1([a,b]) : f(a) = f(b) = 0\}$$

the norms defined by

$$\|f\|_1 = \int_a^b (|f| + |f'|) \quad \text{and} \quad \|f\|_2 = \int_a^b |f'|$$

are equivalents.

12. Prove that on $X = C([0,1])$ the norm $\|.\|_\infty$ and the norm given by $\|f\|_1 = \int_0^1 |f|$ are not equivalent norms.

13. Prove that on $X = C([0,1])$ the norms $\|.\|_1$ and $\|.\|_2$ are not equivalent norms.

14. Show that on ℓ_1 the norms $\|.\|_1$ and $\|.\|_\infty$ are not equivalents.

15. Show that on ℓ_1 the norms $\|.\|_1$ and $\|.\|_2$ are not equivalents.