Exercises 1 – Solutions

① (X,d) metric space, $(x_n),(y_n)$ sequences in X s.t. $x_n \xrightarrow{d} x$, $y_n \xrightarrow{d} y$

$\Rightarrow \lim_{n\to\infty} d(x_n,y_n) = d(x,y)$

$\forall \varepsilon > 0 \quad \exists N_1$ s.t. $\quad d(x_n,x) < \frac{\varepsilon}{2}$ if $n > N_1$

$\exists N_2$ s.t. $\quad d(y_n,y) < \frac{\varepsilon}{2}$ if $n > N_2$

$N := \max\{N_1, N_2\}$

$d(x_n,y_n) \leq d(x_n,x) + d(x,y) + d(y,y_n) \Rightarrow d(x_n,y_n) - d(x,y) \leq d(x_n,x) + d(y_n,y)$

$d(x,y) \leq d(x,x_n) + d(x_n,y_n) + d(y_n,y) \Rightarrow d(x,y) - d(x_n,y_n) \leq d(x,x_n) + d(y_n,y)$

$\Rightarrow \quad |d(x_n,y_n) - d(x,y)| \leq d(x_n,x) + d(y_n,y)$

if $n > N \quad |d(x_n,y_n) - d(x,y)| \leq d(x_n,x) + d(y_n,y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

○!

② $f: \mathbb{R} \to \mathbb{R}$ strictly increasing, $d_f(x,y) := |f(x) - f(y)|$ defines a metric on $\mathbb{R}$.

- $d_f(x,y) \geq 0$ ✓
- $d_f(x,y) = 0 \Leftrightarrow f(x) = f(y) \underset{f \text{ is injective}}{\Longleftrightarrow} x = y$
- $d_f(x,y) = d_f(y,x)$ trivial ✓
- $x,y,z \in \mathbb{R}$

$d_f(x,z) + d_f(z,y) = |f(x) - f(z)| + |f(z) - f(y)| \geq |f(x) - f(z) + f(z) - f(y)|$

$= |f(x) - f(y)| = d_f(x,y)$

○!

③ $X = \{ (x_k)_{k\in\mathbb{N}} : \exists N \in \mathbb{N} \text{ s.t. } x_k = 0 \text{ if } k \geq N \}$

$\Rightarrow$ (X, d_∞) is not complete

$\forall n \in \mathbb{N}$ define

$$x^{(n)}_k := \begin{cases} 1/k & \text{for } k \leq n \\ 0 & \text{for } k > n \end{cases}$$

ie

$$x^{(n)} = (1, \tfrac{1}{2}, \tfrac{1}{3}, \dots, \tfrac{1}{n}, 0, 0, \dots)$$

$\Rightarrow (x^{(n)})_{n\in\mathbb{N}} \subset X$ and

$$d_\infty(x^{(m)}, x^{(n)}) = \sup_{k\in\mathbb{N}} |x^{(m)}_k - x^{(n)}_k| = \sup_{m<k\leq n} \frac{1}{k} = \frac{1}{m+1}$$ ~~sufficiently~~ small for sufficiently large m

$\Rightarrow (x^{(n)})_{n\in\mathbb{N}}$ is Cauchy

If $(x^{(n)})$. is convergent, there exists $x \in X$ s.t.: $\cancel{*}$

$$d_\infty(x^{(n)}, x) \longrightarrow 0 \quad \text{as } n\to\infty$$

In this case $x_k = 1/k$ $k \in \mathbb{N}$, because if $x_k \neq 1/k$ for some k

$$d_\infty(x^{(n)}, x) = \sup_{k\in\mathbb{N}} |x^{(n)}_k - x_k| \geq \left|\frac{1}{k} - x_k\right| > 0$$

so $d_\infty(x^{(n)}, x) \not\to 0$.

But $x = (1, \tfrac{1}{2}, \tfrac{1}{3}, \dots) \notin X$

$\Rightarrow X$ is not complete $\quad$ ↯!

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(5) $H = \{ f \in C[0,1] : 0 = f(0) \leq f(x) \leq f(1), x \in [0,1] \}$
is not compact in $(C[0,1], d_\infty)$.

$A_n := \{ f \in C[0,1] : f(x) > 0 \text{ if } x \in [1-\frac{1}{n}, 1] \} \quad n \in \mathbb{N}$

$\Rightarrow \mathcal{F} := \{ A_n \subset C[0,1] : n \in \mathbb{N} \}$ is an open covering set of H,
but it hasn't got a finite subcovering set. !

(5) $\mathcal{P} := \{ f : \mathbb{R} \to \mathbb{R} : f \text{ is polinomial} \}$

a) $f(x) = \sum_{k=0}^{n} a_k x^k \Rightarrow \|f\| = \max_k \{|a_k|\}$ is a norm on $\mathcal{P}$

- $\|f\| \geq 0$ ✓
- $\|f\| = 0 \iff \max_k |a_k| = 0 \iff a_k = 0 \quad \forall k = 1, \dots, n$
$\iff f(x) = 0$ ✓

- $f(x) = \sum_k a_k x^k$
$g(x) := \sum_\ell b_\ell x^\ell$

$\|f+g\| = \max_k \{ |a_k + b_k| \} \leq \max_k \{ |a_k| + |b_k| \} \leq \max_k |a_k| + \max_k |b_k|$
$= \|f\| + \|g\|$ ✓

b) $f \in \mathcal{P}$, $\|f\| := \sum_{n=1}^{\infty} \frac{|f(n)|}{n!}$ norm on $\mathcal{P}$

$f \in \mathcal{P} \Rightarrow \exists m \in \mathbb{N}, a_0, \dots, a_m \in \mathbb{R} \quad f(x) = \sum_{k=0}^{m} a_k x^k$

If for $K > 0 \quad \sum_{k=0}^{m} |a_k| < K \Rightarrow \forall x \geq 1$:

$$|f(x)| \leq \sum_{k=0}^{m} |a_k| x^k \leq \left(\sum_{k=0}^{m} |a_k| \right) x^m \leq K x^m$$

5) $\Rightarrow$ $\displaystyle\sum_{n=1}^{\infty}\left(\frac{K n^n}{n!}\right) \leftarrow (*)$ is convergent (check by the n-th root test)

$$\Downarrow$$

$\displaystyle\sum_{n=1}^{\infty}\left(\frac{|f(n)|}{n!}\right)$ is convergen ($(*)$ majorizes it)

$\Rightarrow$ $0 \leq \|f\| < \infty$

The all properties of norms come from the properties of series . !

(6) $x=(x_1,x_2),\ y=(y_1,y_2) \in \mathbb{R}^2$

$$d(x,y)=\begin{cases} 1 & \text{if } x_1 \neq y_1 \\ \min\{1, |x_2-y_2|\} & \text{if } x_1 = y_1 \end{cases}$$

$\Rightarrow$ d is not induced by any norm.

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d is not ~~shift invariant~~ homogeneous :

for $x=(0,0),\ y:=(0,1),\ \lambda:=2$

$d(\lambda x, \lambda y) = d((0,0),(0,2)) = \min\{1, |0-2|\} = 1$

and $|\lambda|\, d(x,y) = 2 d((0,0),(0,1)) = 2\cdot \min\{1, |0-1|\} = 2$

o!

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(7) $A = \{x = (x_n)_{n\in\mathbb{N}} \in \ell_2 : |x_n| \leq \frac{1}{\sqrt{n}}, n \in \mathbb{N}\}$

is not compact in ℓ_2.

$$e_n := (0, 0, \ldots, 0, \underset{\substack{\uparrow \\ n\text{ th place}}}{1}, 0, \ldots)$$

$$x^{(n)}_k := \sum_{k=1}^{n} \frac{1}{\sqrt{k}} e_k \implies (x^{(n)}_k)_{k\in\mathbb{N}} \in A \subset \ell_2 \quad \forall n \in \mathbb{N}$$

$$\left(x^{(n)}_k = \begin{cases} 1/k & \text{if } k \leq n \\ 0 & \text{if } k > n \end{cases} \right)$$

$$d_2^2(x^{(n)}, 0) = \|x^{(n)} - 0\|_2^2 = \sum_{k=1}^{n} |x^{(n)}_k|^2 = \sum_{k=1}^{n} \frac{1}{k} \xrightarrow[n\to\infty]{} \infty$$

$\implies$ A is not bounded $\implies$ A is not compact. $\circ$!

(8) $p \in (1, \infty)$, $\frac{1}{p} + \frac{1}{q} = 1$. $x = (x_n)_{n\in\mathbb{N}} \in \ell_p \implies (|x_n|^{p-1})_{n\in\mathbb{N}} \in \ell_q$

$$\frac{1}{p} + \frac{1}{q} = 1 \implies q = \frac{p}{p-1}$$

$$\sum_{n=1}^{\infty} \left(|x_n|^{p-1}\right)^q = \sum_{n=1}^{\infty} |x_n|^{(p-1)\frac{p}{p-1}} = \sum_{n=1}^{\infty} |x_n|^p < \infty$$

$$\Downarrow$$

$$(|x_n|^{p-1})_{n\in\mathbb{N}} \in \ell_q \quad \circ!$$

6/ (9) a) c is closed subspace in l_∞

$c \subset l_\infty$ is clear

for $x = (x_k)_{k\in\mathbb{N}} \in l_\infty$ let $x^{(n)} = (x^{(n)}_k)_{k\in\mathbb{N}} \in c$ s.t.

$$\lim_{n\to\infty} \|x^{(n)} - x\|_\infty = 0 \qquad (*)$$

x is bounded by the Bolzano-Weierstrass theorem it has a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$: $x_{n_k} \xrightarrow[k\to\infty]{} \xi$ $(**)$

If $\xi_n := \lim_{k\to\infty} x_k^{(n)}$ $n\in\mathbb{N}$, then $\xi_n = \lim_{m\to\infty} x^{(n)}_{n_m}$ $(***)$

$(*) \Rightarrow \forall \varepsilon > 0\ \exists N\in\mathbb{N}$ s.t. for $n \geq N$

$$\|x^{(n)} - x\|_\infty < \frac{\varepsilon}{3}$$

$$\Downarrow$$

$$|x^{(n)}_k - x_k| < \frac{\varepsilon}{3} \quad k\in\mathbb{N}$$

$(**) \Rightarrow \exists M\in N$ s.t. for $m \geq N$:

$$|x_{n_m} - \xi| < \frac{\varepsilon}{3}$$

$(***) \Rightarrow \forall n\in\mathbb{N}\ \exists \ell\in\mathbb{N},\ n_\ell \geq M$:

$$|x^{(n)}_{n_\ell} - \xi_n| < \frac{\varepsilon}{3}$$

$\Rightarrow \forall n \geq N$

$$|\xi_n - \xi| \leq |\xi_n - x^{(n)}_{n_\ell}| + |x^{(n)}_{n_\ell} - x_{n_\ell}| + |x_{n_\ell} - \xi| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

i.e. $\lim_n \xi_n = \zeta \implies x$ cannot be divergent

<u>indeed</u>: if x is divergent, it has $\xi \neq \eta \in \mathbb{K}$ accumulation point and $\lim_n \xi_n = \eta$: $\lightning$!

(b)–(5): were showed in the lectures or You can find in any Functional Analysis textbook.

(10) $0 < p < q < \infty$ ℓ_p is dense in ℓ_q.

$x = (x_n) \in \ell_q \implies \forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t.

$$\sum_{n=N+1}^{\infty} |x_n|^q < \varepsilon^q$$

$\implies$ For $u := (x_1, \dots, \cancel{x}_N, 0, 0, \dots) \in \ell_p \subset \ell_q$

$\|x - u\|_q < \varepsilon \implies \ell_p$ is dense in ℓ_q !

8/ (11) $X = \{ f \in C^1([a,b]) : f(a) = f(b) = 0 \}$

$$\|f\|_1 := \int_a^b (|f| + |f'|) \quad , \; \|f\|_2 = \int_a^b |f'|$$

$\Rightarrow$ $\|.\|_1$ and $\|.\|_2$ are equivalents.

$\forall f \in X \quad \|f\|_1 \geq \|f\|_2$ ✓

$$\|f\|_1 - \|f\|_2 = \int_a^b |f| = \int_a^b \Big(\Big| \underbrace{\int_a^t f'(s)\, ds}_{[f(s)]_a^t = f(t) - f(a) = f(t)} \Big| \Big) dt \leq \int_a^b \Big(\int_a^b |f'(s)|\, ds \Big) dt \;(=)$$

$$(=) \int_a^b |f'(s)|\, ds \cdot \int_a^b dt = \|f\|_2 \cdot (b-a)$$

$$\Rightarrow \quad \|f\|_1 \leq (b-a+1) \|f\|_2 \qquad \Rightarrow \|.\|_1 \cong \|.\|_2$$

$\circ$!

(12) $X = C[0,1] \Rightarrow \|.\|_\infty$ and $\|f\|_1 = \int_0^1 |f|$ are not equivalents

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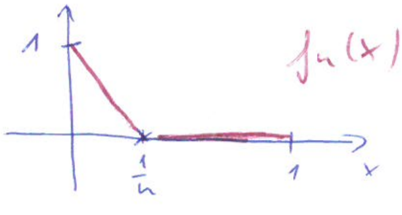
$$f_n(x) := \begin{cases} 1 - nx & \text{if } x \in [0, \frac{1}{n}] \\ 0 & \text{if } x \in [\frac{1}{n}, 1] \end{cases}$$



$\Rightarrow \|f_n\|_\infty = 1 \quad \forall n \in \mathbb{N}$

$$\|f_n\|_1 = \int_0^{1/n} (1 - nx)\, dx = \Big[x - \frac{nx^2}{2} \Big]_0^{1/n} = \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n} \to 0 \quad \text{as } n \to \infty$$

$\Rightarrow \nexists K_1, K_2 \quad K_1 \|f\|_1 \geq \|f\|_2 \geq K_2 \|f\|_1 \quad \forall f$

$\circ$!

9)

(13) $X = C[0,1]$ $\quad \|.\|_1 \not\sim \|.\|_2$

$f_n(x) := x^n \quad x \in [0,1]$

$$\|f_n\|_1 = \int_0^1 |f_n| = \int_0^1 x^n \, dx = \left[\frac{x^{n+1}}{n+1}\right]_0^1 = \frac{1}{n+1}$$

$$\|f_n\|_2 = \sqrt{\int_0^1 |f_n|^2} = \sqrt{\underbrace{\int_0^1 x^{2n} \, dx}_{\left[\frac{x^{2n+1}}{2n+1}\right]_0^1 = \frac{1}{2n+1}}} = \frac{1}{\sqrt{2n+1}}$$

$$\Rightarrow \quad \frac{\|f_n\|_2}{\|f_n\|_1} = \frac{n+1}{\sqrt{2n+1}} \xrightarrow[n\to\infty]{} \infty \qquad \circ!$$

(14) $X = \ell_1$ $\quad \|.\|_1 \not\sim \|.\|_\infty$

$\ell_1 \subset \ell_\infty \qquad \|x\|_1 = \sum_{n=1}^\infty |x_n| \geq \sup_n |x_n| = \|x\|_\infty \qquad x \in \ell_1$

$$x_k^{(n)} := \begin{cases} 1 & \text{if } k \leq n \\ 0 & \text{if } k > n \end{cases} \qquad x^{(n)} = (\underbrace{1, 1, \dots, 1}_{n}, 0, 0, \dots) \in \ell_1$$

$\cancel{x = (x_k^{(n)})_{n\in\mathbb{N}}}$

$$\|x^{(n)}\|_1 = n \xrightarrow[n\to\infty]{} \infty$$

$$\|x^{(n)}\|_\infty = 1 \quad \forall n \in \mathbb{N} \qquad \circ!$$

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(15) On ℓ_1 $\|\cdot\|_1 \not\sim \|\cdot\|_2$

$\ell_1 \subset \ell_2$

$$x_k^{(n)} := \begin{cases} 1/k & \text{if } k \leq n \\ 0 & k > 0 \end{cases}$$

$$x^{(n)} = \left(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, 0, \dots\right) \in \ell_1$$

$$\|x^{(n)}\|_1 = \sum_{k=1}^{n} \frac{1}{k} \xrightarrow[n\to\infty]{} \infty$$

$$\|x^{(n)}\|_2 = \sqrt{\sum_{k=1}^{n} \frac{1}{k^2}} \xrightarrow{n\to\infty} \frac{\pi}{\sqrt{6}}$$

$\circ$!