Functional Analysis,
Exercises 2.

1. Let $(X, \|.\|)$ be a normed space and let M be a closed subspace of it. Show that the expression
$$\|\tilde{x}\| = \operatorname{dist}(x, M) = \inf_{m \in M} \|x - m\|$$
defines a norm on the quotient space $\tilde{X} = X/M$, where $\tilde{x} \in \tilde{X}$ is a representative of the equivalence class of $x \in X$.

2. Let X, Y be normed space over the same field $\mathbb{K}$. For a linear map $T : X \to Y$ we defined the operator norm of T by
$$\|T\| = \sup\{\|Tx\|_Y : \|x\|_X = 1, x \in X\}.$$
Show that $\|.\|$ defines really a norm.

3. Let define for all $f \in L^2([0, a])$ $(a > 0)$ the operator
$$(Af)(x) := xf(x), \quad (x \in [0, a]).$$
Show that A is a bounded linear operator on $L^2([0, a])$ and compute its norm.

4. Let $\varphi \in C([a, b])$ be fixed and consider the operator
$$A_\varphi : C([a, b]) \to \mathbb{K}, \quad A_\varphi f := \int_a^b f\varphi, \quad (f \in C([a, b])).$$
Show that A_φ is a bounded linear operator and compute its norm. Here $C([a, b])$ is considered with $\|.\|_\infty$ norm.

5. Let denote $X = (L^1([0, 1]), \|.\|_{L^1})$ and $Y = (c_0, \|.\|_\infty)$. We define an operator by
$$A : X \to Y, \quad (Au)_n = \int_0^1 u(s)s^n \, \mathrm{d}\, s \quad (n \in \mathbb{N}).$$
Show that A is a bounded linear operator and compute its operator norm.

6. For a fixed sequence $(c_n)_{n\in\mathbb{N}} \in \ell_\infty$ we define an operator for all $u = (u_n)_{n\in\mathbb{N}} \in \ell_p$, ($p \in [1,\infty]$) by
$$A_c u := (c_n u_n)_{n\in\mathbb{N}}.$$
Show that $A_c \in \mathcal{B}(\ell_p)$ and give its norm.

7. Denote $X = C([0,1])$ with the $\|.\|_\infty$ norm. Show that for a fixed $\varphi \in X$
$$A_\varphi : X \to X, \quad A_\varphi f = \varphi f$$
is a bounded linear operator and give its norm.

8. Denote $X = C([0,1])$ with the $\|.\|_\infty$ norm. Define the following operators on X:
$$(Au)(x) = xu(x), \quad (Bu)(x) = x\int_0^1 u(t)\,\mathrm{d}\,t \quad (u \in X, x \in [0,1]).$$
Show that $A, B \in \mathcal{B}(X)$ and compute the operator norms of the composition operators AB and BA.

9. Denote $X = C([0,1])$ with the $\|.\|_\infty$ norm. Show that the integral operator given by
$$(Af)(t) = \int_0^\pi \cos(ts)f(s)\,\mathrm{d}\,s \quad (f \in X, t \in [0,\pi])$$
defines a bounded linear operator on X, i.e. $A \in \mathcal{B}(X)$ and give its norm.

10. Let $m, n \in \mathbb{N}$ and consider the vector spaces $X = \mathbb{K}^m$ and $Y = \mathbb{K}^n$. Let M be a fixed n by m matrix over $\mathbb{K}$, i.e. $M \in \mathcal{M}_{nm}(\mathbb{K})$ and define the operator
$$A : X \to Y, \quad Ax := Mx = \left[\sum_{k=1}^m M_{ik}x_k\right]_{i\in\{1,2,\ldots,n\}}.$$
Compute the operator norm of A when
   (a) $\|.\|_X = \|.\|_Y = \|.\|_\infty$,
   (b) $\|.\|_X = \|.\|_Y = \|.\|_1$,
   (c) $\|.\|_X = \|.\|_Y = \|.\|_2$.

   Let $X = Y = \mathbb{R}^2$ and give $\|A\|$ in the cases above if
$$M = \begin{pmatrix} a & a \\ b & b \end{pmatrix}, \quad a, b > 0.$$