Exercises 2. – Solutions

(1) $(X, \|.\|)$ normed space, $M \subset X$ closed subspace

$\Rightarrow$ $\|\tilde{x}\| = \text{dist}(x, M) = \inf_{m \in M} \|x - m\|$ defines a norm on $\tilde{X} = X/M$.

- $\|.\|$ is well defined:

$\forall x, y \in X$ with $x - y \in M$

$$\text{dist}(x, M) = \inf\{\|x-m\| : m \in M\} = \inf\{\|y - \underbrace{(y-x+m)}_{\in M}\| : m \in M\} =$$

$$= \inf\{\|y-m\| : m \in M\} = \text{dist}(y, M) \quad \checkmark$$

- ~~$\|\tilde{x}\| = 0 \Leftrightarrow x \in M \Leftrightarrow \tilde{x} = 0$~~

- $\|\alpha \tilde{x}\| = |\alpha| \cdot \|\tilde{x}\|$ is trivial

- $$\|\tilde{x} + \tilde{y}\| = \inf\{\|x+y-m\| : m \in M\} = \inf\{\|x - m_x + y - m_y : m_x, m_y \in M\}$$

$$\leq \inf\{\|x - m_x\| : m_x \in M\} + \inf\{\|y - m_y\| : m_y \in M\}$$

$$= \|\tilde{x}\| + \|\tilde{y}\|$$

- $\tilde{x} = 0 \Rightarrow \|\tilde{x}\| = 0$ trivial

Assume that $\|\tilde{x}\| = 0$, we have to show that $x \in M$.

By the definition of dist, there exists a sequence $(m_n)_{n \in \mathbb{N}} \subset M$

s.t. $\|x - m_n\| \xrightarrow[n \to \infty]{} 0$ ie $m_n \xrightarrow{\|.\|} x$. As M is closed

$$\Downarrow$$

$$x \in M$$

. !

2/ ② X, Y normed space over $\mathbb{K}$, $T: X \to Y$ linear

$$\|T\| := \sup\{\|Tx\|_Y : \|x\|_X = 1,\ x \in X\}$$

Show that $\|.\|$ defines a norm.

The only non-trivial property is the triangle inequality:

$$\|T_1 + T_2\|_Y = \sup_{\|x\|_X=1} \|(T_1+T_2)(x)\|_Y \leq \sup_{\|x\|=1}\left(\|T_1 x\|_Y + \|T_2 x\|_Y\right) \leq$$

$$\leq \sup_{\|x\|_X=1} \|T_1 x\|_Y + \sup_{\|x\|_X=1} \|T_2 x\|_Y = \|T_1\| + \|T_2\| \quad .!$$

③ $f \in L^2([0,a])$, $(Af)(x) := x f(x) \quad x \in [0,a]$

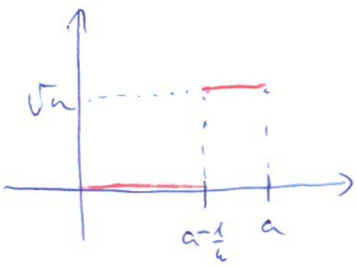
$\|A\| = ?$

For any $f \in L^2([0,a])$

$$\|Af\|_2^2 = \int_0^a x^2 |f(x)|^2 dx \leq a^2 \int_0^a |f(x)|^2 dx = a^2 \|f\|_2^2$$

$\Rightarrow$ A is bounded and $\|A\| \leq a$.

Define the sequence $f_n(x) := \begin{cases} \sqrt{n} & \text{if } a - \frac{1}{n} \leq x \leq a \\ 0 & \text{otherwise} \end{cases}$



$\Downarrow$

$$\|f_n\|_2^2 = \int_{a-\frac{1}{n}}^a |\sqrt{n}|^2 dx = n\left(a - a + \frac{1}{n}\right) = 1$$

$\Downarrow$

$\|f_n\|_2 = 1$

$$\|Af_n\|_2^2 = \int_{a-\frac{1}{n}}^a x^2 n = \frac{n}{3}\underbrace{\left(a^3 - \left(a-\frac{1}{n}\right)^3\right)}_{\frac{3a^2}{n} - \frac{3a}{n^2} + \frac{1}{n^3}} = \left(a^2 - \frac{a}{n} + \frac{1}{3n^2}\right) \xrightarrow[n\to\infty]{} a^2 \Rightarrow \|A\| \geq a$$

$\Downarrow$

$\boxed{\|A\| = a}$

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(4) $\varphi \in C[a,b]$ fixed

$$A_\varphi : C[a,b] \to \mathbb{K}, \quad A_\varphi f := \int_a^b f\varphi \qquad f \in C[a,b]$$

$\|A_\varphi\| = ?$

- $A_\varphi : C[a,b] \to \mathbb{K}$ is linear:

$$A_\varphi(\alpha f + \beta g) = \int_a^b (\alpha f + \beta g)\varphi = \alpha \int_a^b f\varphi + \beta \int_a^b g\varphi = \alpha A_\varphi f + \beta A_\varphi g \checkmark$$

- $\forall f \in C[a,b]$:

$$|A_\varphi f| = \left|\int_a^b f\varphi\right| \leq \int_a^b |f||\varphi| \leq \int_a^b \|f\|_\infty \cdot |\varphi| = \|f\|_\infty \int_a^b |\varphi|$$

$\Rightarrow A_\varphi$ is bounded and $\|A_\varphi\| \leq \int_a^b |\varphi|$

For any $\varepsilon > 0$ let define

$$f_\varepsilon := \frac{\overline{\varphi}}{|\varphi| + \varepsilon}$$

$\Rightarrow f_\varepsilon \in C[a,b]$ and $\|f_\varepsilon\|_\infty \leq 1$

$$|A_\varphi f_\varepsilon| = \int_a^b \frac{|\varphi|^2}{|\varphi| + \varepsilon} \geq \int_a^b \frac{|\varphi|^2 - \varepsilon^2}{|\varphi| + \varepsilon} = \int_a^b (|\varphi| - \varepsilon) =$$

$$= \int_a^b |\varphi| - \varepsilon(b-a)$$

$\Rightarrow \|A_\varphi\| = \sup\{|A_\varphi f| : \|f\|_\infty \leq 1\} \geq \sup\{|A_\varphi f_\varepsilon| : \varepsilon > 0\} \geq \int_a^b |\varphi|$

$\Rightarrow \|A\| = \int_a^b |\varphi|$!

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(5) $X = L^1[0,1]$, $Y = (c_0, \|\cdot\|_\infty)$

$A: X \to Y$, $(Au)_n := \int_0^1 u(s) s^n ds$ $n \in \mathbb{N}$.

$\|A\| = ?$

- A is linear - trivial $\Rightarrow A \in \mathcal{L}(L^1[0,1], c_0)$

- For any $u \in L^1[0,1]$

$$\|Au\|_\infty = \sup\left\{\left|\int_0^1 u(s) s^n ds\right| : n \in \mathbb{N}\right\} \leq$$

$$\leq \left|\sup\left\{\underbrace{\int_0^1 |u(s)| 1^n ds}_{\|u\|_1} : n \in \mathbb{N}\right\}\right| = \|u\|_1$$

$\Rightarrow \|A\| \leq 1$

Let $v(x) = 1$, $x \in [0,1] \Rightarrow \|v\|_1 = \int_0^1 |v(x)| dx = 1 \Rightarrow v \in L^1[0,1]$

$$\|A\| = \sup\{\|Au\|_\infty : \|u\|_1 = 1\} \geq \|Av\|_\infty =$$

$$= \sup\left\{\underbrace{\int_0^1 s^n ds}_{\left[\frac{s^{n+1}}{n+1}\right]_0^1 = \frac{1}{n+1}} : n \in \mathbb{N}\right\} = \sup\left\{\frac{1}{n+1} : n \in \mathbb{N}\right\} = 1$$

$\Downarrow$

$\|A\| \geq 1$

$$\Rightarrow \underline{\underline{\|A\| = 1}}$$

!

5/ (b) $c=(c_n)_{n\in\mathbb{N}} \in \ell_\infty$ fixed

$A_c u := (c_n u_n)_{n\in\mathbb{N}} \quad u=(u_n)_{n\in\mathbb{N}} \in \ell_p \quad p\in[1,\infty]$

Show that $A_c \in \mathcal{B}(\ell_p)$, $\|A\|=?$

- For $p\in[1,\infty)$, $\forall\ x=(x_n)_{n\in\mathbb{N}}\in\ell_p$

$$\|Ax\|_p = \left(\sum_{k=1}^{n} |c_n x_n|^p\right)^{1/p} \le \Big\{ \underbrace{\sup_{n\in\mathbb{N}} |c_n|^p}_{\|c\|_\infty^p} \cdot \sum_{n=1}^{n} |x_n|^p \Big\}^{1/p} =$$

$$= \|c\|_\infty \cdot \|x\|_p \quad \Rightarrow \|A\| \le \|c\|_\infty$$

For $p=\infty$

$$\|Ax\|_\infty = \sup\{|c_n x_n| : n\in\mathbb{N}\} \le \sup_n |c_n| \cdot \sup_n |x_n| =$$

$$= \|c\|_\infty \cdot \|x\|_\infty$$

$$\hookrightarrow \|A\| < \|c\|_\infty$$

$\Rightarrow p\in[1,\infty]: \|A\| \le \|c\|_\infty$

Let $e_n = (0,0,\dots,0,\underset{\substack{\uparrow \\ n\text{th site}}}{1},0,\dots) \Rightarrow \|e_n\|_p = 1$ for any $p\in[0,\infty]$

$A_c e_n = (0,0,\dots,0,\underset{\substack{\uparrow \\ n\text{th}}}{c_n},0,\dots) \quad n\in\mathbb{N}$

$$\Rightarrow \|A\| = \sup\{\|A_c x\|_p : \|x\|_p \le 1\} \ge \sup\{\|A_c e_n\|_p : n\in\mathbb{N}\} =$$

$$= \sup\{|c_n| : n\in\mathbb{N}\} = \|c\|_\infty \quad \Rightarrow \|A\| \ge \|c\|_\infty$$

$$\Rightarrow \boxed{\|A\| = \|c\|_\infty}$$

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(7) $X = C[0,1]$ with $\|.\|_\infty$, $\varphi \in X$ fixed

$A_\varphi : X \to X$, $A_\varphi f := \varphi f$

$\|A_\varphi\| = ?$

- A_φ is linear ✓
- $\forall\ u \in X$

$\|A_\varphi u\|_\infty = \|\varphi u\|_\infty = \sup\{|\varphi(x)u(x)| : x \in [0,1]\} \leq$

$\leq \sup\{|\varphi(x)| : x \in [0,1]\} \cdot \sup\{|u(x)| : x \in [0,1]\} =$

$= \|\varphi\|_\infty \cdot \|u\|_\infty \Rightarrow \|A_\varphi\| \leq \|\varphi\|_\infty$ A_φ bounded ✓

For $a(x) := 1$ $x \in [0,1]$ $\Rightarrow \|a\|_\infty = 1$, $a \in X$

$\|A\| = \sup\{\|A_\varphi u\| : \|u\|_\infty \leq 1\} \geq \|A_\varphi a\|_\infty = \|\varphi\|_\infty \Rightarrow \|A\| \geq \|\varphi\|_\infty$

$\Rightarrow$ $\underline{\underline{\|A_\varphi\| = \|\varphi\|_\infty}}$!

(8) $X = (C[0,1], \|.\|_\infty)$, ~~$A : X \to X$~~

$(Au)(x) = x\,u(x)$, $(Bu)(x) := x \int_0^1 u(t)\,dt$ $u \in X$, $x \in [0,1]$.

Show that $A, B \in \mathcal{B}(X)$, $\|AB\|, \|BA\| = ?$

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(7) $\Rightarrow \|A\| = 1$ and similarly $\|B\| = 1$

$u \in X$, $x \in [0,1]$

$(ABu)(x) = x^2 \int_0^1 u(t)\,dt$, $(BAu)(x) = x \int_0^1 t\,u(t)\,dt$

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$$\|ABu\|_\infty = \sup\left\{ \left| x^2 \int_0^1 u(t)\,dt \right| : x \in [0,1] \right\} =$$

$$= \left| \int_0^1 u(t)\,dt \right| \leq \int_0^1 |u(t)|\,dt \leq \int_0^1 \|u\|_\infty\,dt = \|u\|_\infty$$

$$\Rightarrow AB \in \mathcal{B}(X) \quad , \quad \|AB\| \leq 1$$

$$\|BAu\|_\infty = \sup\left\{ \left| x \int_0^1 t u\,dt \right| : x \in [0,1] \right\} =$$

$$= \left| \int_0^1 t u(t)\,dt \right| \leq \int_0^1 t\,|u(t)|\,dt \leq \|u\|_\infty \underbrace{\int_0^1 t\,dt}_{\left[\frac{t^2}{2}\right]_0^1 = \frac{1}{2}} = \|u\|_\infty \cdot \frac{1}{2}$$

$$\Rightarrow BA \in \mathcal{B}(X) \,, \quad \|BA\| \leq \frac{1}{2}$$

For $a(x) = 1$, $x \in [0,1]$ $\quad a \in X$ with $\|a\|_\infty = 1$

$$\|AB\| = \sup\left\{ \|ABu\|_\infty : \|u\|_\infty \leq 1 \right\} \geq \|ABa\|_\infty =$$

$$= \sup\left\{ \left| x^2 \int_0^1 1\,dt \right| : x \in [0,1] \right\} = 1 \qquad \Rightarrow \quad \|AB\| \geq 1$$

$$\Downarrow$$

$$\boxed{\|AB\| = 1}$$

$$\|BA\| = \sup\left\{ \|BAu\|_\infty : \|u\|_\infty \leq 1 \right\} \geq$$

$$\geq \|BAa\|_\infty = \sup\left\{ \left| x \int_0^1 t\,dt \right| : x \in [0,1] \right\} = \int_0^1 t\,dt = \frac{1}{2}$$

$$\Downarrow$$

$$\|BA\| \geq \frac{1}{2}$$

$$\Downarrow$$

$$\boxed{\|BA\| = \frac{1}{2}}$$

∘ !

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(9) $X = (C[0,1], \|.\|_\infty)$, $(Af)(t) = \int_0^\pi \cos(ts) f(s) ds$ $f \in X$

$t \in [0,\pi]$

$\Rightarrow$ $A \in \mathcal{B}(X)$, $\|A\| = ?$

- A is linear (trivial) ✓

- $\forall$ $t \in [0,\pi]$

$$|(Af)(t)| = \left| \int_0^\pi \cos(ts) f(s) ds \right| \leq \int_0^\pi |\cos(ts)| \cdot |f(s)| ds \leq$$

$$\leq \|f\|_\infty \int_0^\pi |\cos(ts)| ds$$

Since $\int_0^\pi |\cos(ts)| ds \leq \int_0^\pi 1 ds = \pi$ and

for $t=0$ $\int_0^\pi |\cos(0 \cdot s)| ds = \int_0^\pi 1 ds = \pi$, we get

$|(Af)(t)| \leq \|f\|_\infty \cdot \pi$ $\Rightarrow$ $A \in \mathcal{B}(X)$ and $\|A\| \leq \pi$

- For $\varphi(t) = 1$ $t \in [0,\pi]$ $\Rightarrow$ $\|\varphi\|_\infty = 1$, $\varphi \in X$

$\|A\| = \sup \{ |(Af)(t)| : \|f\|_\infty \leq 1 \} \geq \cancel{|(A\varphi)(t)|} \sup \{ |(A\varphi)(t)| : t \in [0,\pi] \}$

$= \sup_{t \in [0,\pi]} \left| \int_0^\pi \cos(ts) ds \right| = \pi$ $\Rightarrow$ $\|A\| \geq \pi$

$\Rightarrow$ $\boxed{\|A\| = \pi}$

o!

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$X = \mathbb{K}^m,\ Y = \mathbb{K}^n,\ A: X \to Y,\ M \in \mathcal{M}_{nm}(\mathbb{K})$

$$Ax := Mx = \left[\sum_{k=1}^{m} M_{ik} x_k\right], \quad \|A\| = ?$$

a) $\|\cdot\|_X = \|\cdot\|_Y = \|\cdot\|_\infty$

$$\|Ax\|_\infty = \max_{i=1,\dots,n} \left|\sum_{j=1}^{m} M_{ij} x_j\right| \le \max_i \sum_{j=1}^{m} |M_{ij}| \cdot |x_j| \le$$

$$\le \left(\max_{i=1,\dots,n} \sum_{j=1}^{m} |M_{ij}|\right) \cdot \underbrace{\left(\max_{j=1,\dots,m} |x_j|\right)}_{\|x\|_\infty}$$

$$\Rightarrow \quad \|A\|_\infty \le \max_{i=1,\dots,n} \sum_{j=1}^{m} |M_{ij}|$$

Suppose that $\max_{i=1,\dots,n} \sum_{j=1}^{m} |M_{ij}| = \sum_{j=1}^{m} |M_{pj}|$

$\Rightarrow$ define $x_j = \operatorname{sgn} M_{pj} \quad j = 1,\dots,m \quad \Rightarrow \|x\|_\infty = 1$

and $\boxed{\|A\| = \max_{i=1,\dots,n} \sum_{j=1}^{n} |a_{ij}|}$

b) $\|\cdot\|_X = \|\cdot\|_Y = \|\cdot\|_1$

$$\|Ax\|_1 = \sum_{i=1}^{n} \left|\sum_{j=1}^{m} M_{ij} x_j\right| \le \sum_{i=1}^{n} \sum_{j=1}^{m} |M_{ij}| \cdot |x_j| =$$

$$= \sum_{j=1}^{m} \left(\sum_{i=1}^{n} |M_{ij}|\right) |x_j| \le \sum_{j=1}^{m} \left(\max_{1 \le k \le m} \sum_{i=1}^{n} |M_{ik}|\right) |x_j|$$

$$= \left(\max_{1 \le j \le m} \sum_{i=1}^{n} |M_{ij}|\right) \left(\sum_{j=1}^{m} |x_j|\right) = \left(\max_{1 \le j \le m} \sum_{i=1}^{n} |M_{ij}|\right) \cdot \|x\|_1$$

$$\Rightarrow \quad \|A\| \leq \max_{1\leq j\leq m} \sum_{i=1}^{n} |M_{ij}|$$

If $\max_{1\leq j\leq m} \sum_{i=1}^{n} |M_{ij}| = \sum_{i=1}^{n} |M_{ip}|$ for some p

let define $x = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in p \rightsquigarrow \|x\|_1 = 1$

and $\|Ax\|_1 = \sum_{i=1}^{m} |M_{ip}|$

$$\Rightarrow \boxed{\|A\| = \max_{1\leq j\leq m} \sum_{i=1}^{n} |M_{ij}|}$$

c) For $M \in \mathcal{M}_{nm}$ $(M^*M)^* = M^*M$ selfadjoint matrix

$M^*M \in \mathcal{M}_{mm}$

$\Rightarrow \exists \lambda \in \mathbb{R},\ x \in \mathbb{K}^m$ st

$(M^*M)x = \lambda x$ (eigenequation)

$\Leftrightarrow x^*(M^*M)x = \lambda x^*x$

$\Leftrightarrow (Mx)^*(Mx) = \lambda x^*x$

$$\Rightarrow \lambda = \frac{(Mx)^*(Mx)}{x^*x} \geq 0$$

11)

Let denote $x_1, x_2, \dots, x_n$ the eigenvectors of M^*M, they form an orthonormal basis

$$\Downarrow$$

$$\forall x \in \mathbb{K}^n \quad x = \sum_{i=1}^{n} \alpha_i x_i$$

$$\frac{\|Mx\|_2^2}{\|x\|_2^2} = \frac{(Mx)^*(Mx)}{x^*x} = \frac{x^*M^*Mx}{x^*x} = \frac{\left(\sum_i \bar{\alpha}_i x_i^*\right)M^*M\left(\sum_i \alpha_i x_i\right)}{\left(\sum_i \bar{\alpha}_i x_i^*\right)\left(\sum_i \alpha_i x_i\right)}$$

$$\Downarrow$$

$$= \frac{\sum_i |\alpha_i|^2 \lambda_i}{\sum_i |\alpha_i|^2} \leq \max_i \lambda_i$$

$$\uparrow$$

$$x_i^* x_j = \delta_{ij}$$

$$\Longrightarrow \quad \|M\|^2 = \sup_{x \neq 0} \frac{\|Mx\|_2^2}{\|x\|_2^2} \leq \max_i \lambda_i$$

If the maximum is taken for $i = p$ $\Longrightarrow$ for $x = x_p$

we get $\|A\|_2 = \max_i \left[\underbrace{\lambda_i(M^*M)}_{\text{the eigenvalues of } M^*M}\right]^{1/2}$

o!

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11 $a, b > 0$ $M = \begin{pmatrix} a & a \\ b & b \end{pmatrix}$

$\Rightarrow$ a) $\|A\| = 2\max\{a, b\}$

b) $\|A\| = a+b$

c) $\|A\| = \sqrt{2}\sqrt{a^2+b^2}$

o!