Functional Analysis,
Exercises 3.

1. Prove that
$$f(x) = x_1 - 3x_2, \quad (x = (x_n)_{n\in\mathbb{N}} \in \ell_2)$$
defines a bounded linear functional on ℓ_2 and compute its norm.

2. Compute the norm of the $\phi \in (C([0,1]), \|.\|_\infty)^*$ functional if
   (a) $\phi(u) = \int_0^1 xu(x) \,\mathrm{d}\, x$, $u \in C([0,1])$
   (b) $\phi(u) = \int_0^1 x^2 u(x) \,\mathrm{d}\, x$, $u \in C([0,1])$
   (c) $\phi(u) = \int_0^1 \sin(2\pi x)u(x) \,\mathrm{d}\, x$, $u \in C([0,1])$
   (d) $\phi(u) = \int_0^1 \sqrt{x}u(x) \,\mathrm{d}\, x$, $u \in C([0,1])$

3. Compute the norm of the $\phi \in (C([-1,1]), \|.\|_\infty)^*$ functional if
   (a) $\phi(u) = \int_0^1 u(x) \,\mathrm{d}\, x$, $u \in C([-1,1])$
   (b) $\phi(u) = \int_{-1}^1 \operatorname{sgn}(x)u(x) \,\mathrm{d}\, x$, $u \in C([-1,1])$
   (c) $\phi(u) = \int_{-1}^1 u(x) \,\mathrm{d}\, x - u(0)$, $u \in C([-1,1])$

4. Let f be a positive linear functional on $(C([a,b]), \|.\|_\infty)$. Show that $\|f\| = f(\chi_{[a,b]})$, where $\chi_{[a,b]}$ is the characteristic function of $[a,b]$.

5. Show that $(\mathbb{K}^n, \|.\|_1)^* = (\mathbb{K}^n, \|.\|_\infty)$.

6. Let $x = (x_n)_{n\in\mathbb{N}} \in \mathfrak{c}$, where $\mathfrak{c}$ denotes the linear space on convergent sequences with the $\|.\|_\infty$ norm. Denote $(e_n)_{n\in\mathbb{N}}$ the standard basis on $\mathfrak{c}$, $\alpha(x) := \lim_n x_n$ and $e := (1, 1, \dots)$. Show that for any $\phi \in \mathfrak{c}^*$ the following statments are true.
   (a) $(\phi(e_n))_{n\in\mathbb{N}} \in \ell_1$,
   (b) $\phi(x) = \alpha(x)\left(\phi(e) - \sum_{n=1}^\infty \phi(e_n)\right) + \sum_{n=1}^\infty x_n\phi(e_n)$,
   (c) $\|\phi\| = |\phi(e) - \sum_{n=1}^\infty \phi(e_n)| + \sum_{n=1}^\infty |\phi(e_n)|$.

7. Prove that $\mathfrak{c}^* = \ell_1$.

8. Let X be a normed space. Show that if for some $x \in X$ $\phi(x) = 0$ holds for all $\phi \in X^*$ then $x = 0$.

9. Let X be a normed space. Show that if for some $x, y \in X$ $\phi(x) = \phi(y)$ holds for all $\phi \in X^*$ then $x = y$.

10. Let X be a normed space. Show that for any $x, y \in X$, $x \neq y$ there exists $\phi \in X^*$ s.t.

$$\|\phi\| = 1 \quad \text{and} \quad \phi(x) \neq \phi(y).$$

11. Let f be a linear functional on a normed space X, (i.e. $f \in X'$ the algebraic dual). Show that f is continous (i.e. $f \in X^*$ the topological dual) iff $\ker f = \{x \in X : f(x) = 0\}$ is a closed linear subspace in X.