Exercises 3 – Solutions

① $f(x) = x_1 - 3x_2$, $x = (x_n)_{n\in\mathbb{N}} \in l_2$

Prove f is a bounded lin. functional on l_2 and compute its norm.

We know that $(l_2)^* = l_2$ and for any $\varphi \in (l_2)^*$ $\exists!$

$v \in l_2$ st $\varphi(x) = \sum_{i=1}^{n} x_i v_i$, and $\|\varphi\| = \|v\|_2$.

↪ Let define $v = (1, -3, 0, \dots) \in l_2$, then

$$f(x) = x_1 - 3x_2 = \sum_{i=1}^{\infty} x_i v_i \quad \text{and}$$

$$\|f\| = \|v\|_2 = \sqrt{1^2 + (-3)^2} = \sqrt{10}$$

.!

② Compute the norm of $\varphi \in (C[0,1], \|\cdot\|_\infty)^*$ if $u \in C[0,1]$ and

a) $\varphi(u) = \int_0^1 x\, u(x)\, dx$

For all $u \in C[0,1]$ with $\|u\|_\infty \le 1$

$$|\varphi(u)| = \left|\int_0^1 x\,u(x)\,dx\right| \le \int_0^1 |x\,u(x)|\,dx \le \int_0^1 x \cdot \|u\|_\infty\,dx = \|u\|_\infty \cdot \underbrace{\int_0^1 x\,dx}_{[x^2/2]_0^1 = \frac{1}{2}} = \frac{1}{2}\|u\|_\infty$$

$\Rightarrow \|\varphi\| \le \frac{1}{2}$.

For $u(x) = 1$, $x \in [0,1]$, $\|u\|_\infty = 1$ and

$$\varphi(u) = \int_0^1 x\,dx = \frac{1}{2} \quad \Rightarrow \quad \|\varphi\| \ge \frac{1}{2}$$

$$\Longrightarrow \quad \underline{\underline{\|\varphi\| = \frac{1}{2}}}$$

.!

2) b) $\forall\, u \in C[0,1],\ \|u\|_\infty \le 1$:

$$|\varphi(u)| = \left|\int_0^1 x^2 u(x)\,dx\right| \le \int_0^1 |x^2 u(x)|\,dx \le \int_0^1 x^2 \|u\|_\infty\,dx = \frac{1}{3}\|u\|_\infty$$

$$\Rightarrow \quad \|\varphi\| \le \frac{1}{3}$$

for $u(x)=1 \ \forall x \in [0,1]$, $\|u\|_\infty = 1$ and

$$\varphi(u) = \int_0^1 x^2\,dx = \frac{1}{3} \quad \Rightarrow \quad \|\varphi\| \ge \frac{1}{3}$$

$$\Rightarrow \quad \|\varphi\| = \frac{1}{3}$$

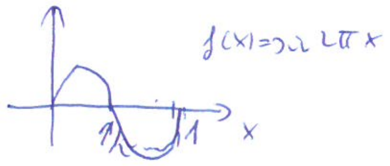
!

c) $\varphi(u) = \int_0^1 \sin(2\pi x)\, u(x)\,dx$

$u \in C[0,1],\ \|u\|_\infty \le 1$

$\hookrightarrow$ $|\varphi(u)| = \left|\int_0^1 \sin(2\pi x) u(x)\,dx\right| \le \int_0^1 |\sin(2\pi x) u(x)|\,dx \le$

$$\le \|u\|_\infty \int_0^1 |\sin(2\pi x)|\,dx$$



$$\Rightarrow \int_0^1 |\sin(2\pi x)|\,dx = \int_0^{1/2} \sin(2\pi x)\,dx - \int_{1/2}^1 \sin(2\pi x)\,dx$$

$$\underset{\text{symmetry}}{=} 2\cdot \int_0^{1/2} \sin 2\pi x\,dx = 2\cdot\left[-\frac{\cos 2\pi x}{2\pi}\right]_0^{1/2} =$$

$$= \frac{1}{\pi}\left(-(-1)-(-1)\right) = \frac{2}{\pi}$$

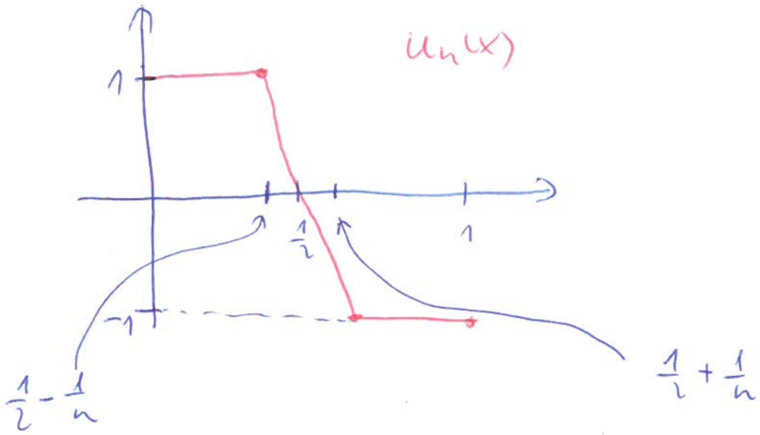
$\hookrightarrow$ $|\varphi(u)| \le \frac{2}{\pi}\|u\|_\infty$

$$\Downarrow$$

$$\|\varphi\| \le \frac{2}{\pi}$$

3)

Let define the sequence $(u_n)_{n\in\mathbb{N}} \subset C[0,1]$ by



$$u_n(x) := \begin{cases} 1 & \text{if } x\in\left[0,\frac{1}{2}-\frac{1}{n}\right] \\ n\left(\frac{1}{2}-x\right) & \text{if } x\in\left[\frac{1}{2}-\frac{1}{n},\frac{1}{2}+\frac{1}{n}\right] \\ -1 & \text{if } x\in\left[\frac{1}{2}+\frac{1}{n},1\right] \end{cases}$$

then $u_n \in C[0,1]$ $\forall n\in\mathbb{N}$ ~~and~~ and $\|u_n\|_\infty = 1$

$$\hookrightarrow \ |\varphi(u_n)| = \left|\int_0^1 \sin(2\pi x)\,u_n(x)\,dx\right| =$$

$$= \left| \int_0^{1/2-1/n} |\sin(2\pi x)|\,dx + \int_{1/2-1/n}^{1/2+1/n} \sin(2\pi x)\,u_n(x)\,dx + \int_{1/2+1/n}^{1} |\sin(2\pi x)|\,dx \right| =$$

$$= \left| \int_0^1 |\sin(2\pi x)|\,dx - \int_{1/2-1/n}^{1/2+1/n} |\sin(2\pi x)|\,dx + \int_{1/2-1/n}^{1/2+1/n} \sin(2\pi x)\,u_n(x)\,dx \right| \underset{\substack{\uparrow \\ \text{integration} \\ \text{by parts}}}{\geq} \dots$$

$$\dots \geq \int_0^1 |\sin(2\pi x)|\,dx - \frac{2}{n} - \frac{2}{n} = \int_0^1 |\sin 2\pi x|\,dx - \frac{4}{n} \underset{\substack{\uparrow \\ \|u_n\|_\infty = 1}}{=}$$

$$= \Big(\underbrace{\int_0^1 |\sin 2\pi x|\,dx}_{\frac{2}{\pi}} - \frac{4}{n}\Big)\|u_n\|_\infty = \left(\frac{2}{\pi} - \frac{4}{n}\right)\|u_n\|_\infty$$

4/

$\Rightarrow$ for any $\varepsilon > 0$, with $n \in \mathbb{N}$ s.t. $\frac{4}{n} < \varepsilon$

we get

$$\frac{|\varphi(u_n)|}{\|u_n\|_\infty} \geq \int_0^1 |\sin(2\pi x)| dx - \varepsilon = \frac{2}{\pi} - \varepsilon$$

$\hookrightarrow \|\varphi\| = \sup \left\{ \frac{|\varphi(u)|}{\|u\|_\infty} : \|u_\infty\| \neq 0 \right\} \geq \frac{2}{\pi}$

$$\Rightarrow \|\varphi\| = \frac{2}{\pi}$$

o!

d) $\varphi(u) = \int_0^1 \sqrt{x}\, u(x)\, dx$

if $u \in C[0,1]$, $\|u\|_\infty \leq 1$

$$|\varphi(u)| = \left| \int_0^1 \sqrt{x}\, u(x) dx \right| \leq \int_0^1 |\sqrt{x}\, u(x)| dx \leq \|u\|_\infty \int_0^1 \sqrt{x} dx = \frac{2}{3} \|u\|_\infty$$

$$\hookrightarrow \|\varphi\| \leq \frac{2}{3}$$

for $u(x) = 1$, $x \in [0,1] \rightsquigarrow \|u\|_\infty = 1$

and $|\varphi(u)| = \left| \int_0^1 \sqrt{x} dx \right| = \frac{2}{3} \rightsquigarrow \|\varphi\| \geq 2/3$

$$\Rightarrow \underline{\underline{\|\varphi\| = \frac{2}{3}}}$$

o!

(3) Compute the norm of $\varphi \in (C[-1,1], \|\cdot\|_\infty)^*$ if

a) $\varphi(u) = \int_0^1 u(x)\,dx$

For any $u \in C[-1,1]$, $\|u\|_\infty \leq 1$

$$|\varphi(u)| = \left|\int_0^1 u(x)\,dx\right| \leq \int_0^1 |u(x)|\,dx \leq \|u\|_\infty \cdot 1$$

$$\Rightarrow \|\varphi\| \leq 1 .$$

For $u(x) = 1$ if $x \in [-1,1]$ $\leadsto \|u\|_\infty = 1$

and $\quad |\varphi(u)| = \int_0^1 dx = 1 \quad \to \|\varphi\| \geq 1$

$\Rightarrow \underline{\|\varphi\| = 1}$ $\qquad$ o!

b) $\varphi(u) = \int_{-1}^1 \operatorname{sgn}(x) u(x)\,dx$.

For all $u \in C[-1,1]$, $\|u\|_\infty \leq 1$

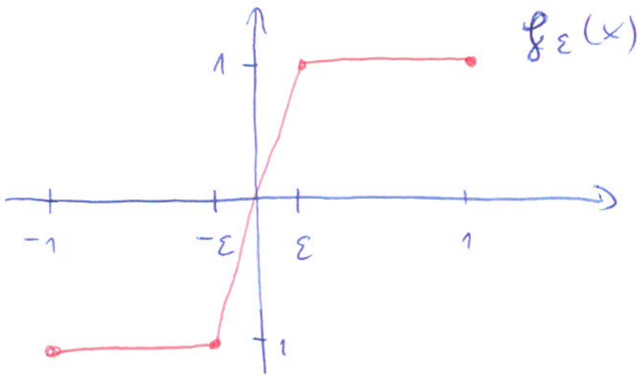
$$|\varphi(u)| = \left|\int_{-1}^1 \operatorname{sgn}(x) u(x)\,dx\right| \leq \int_{-1}^1 \underbrace{|\operatorname{sgn}(x)|}_{=1} \cdot |u(x)|\,dx =$$

$$= \int_{-1}^1 |u(x)|\,dx \leq \|u\|_\infty \cdot \int_{-1}^1 dx = 2 \cdot \|u\|_\infty$$

$$\hookrightarrow \|\varphi\| \leq 2$$

6)

Let define for all $\varepsilon \in (0,1)$



i.e. $$f_\varepsilon(x) := \begin{cases} -1 & , \text{ if } x \in [-1,-\varepsilon] \\ \frac{x}{\varepsilon} & , \text{ if } x \in [-\varepsilon,\varepsilon] \\ 1 & , \text{ if } x \in [\varepsilon, 1] \end{cases}$$

$\hookrightarrow$ $f_\varepsilon \in C[-1,1]$ and $\|f_\varepsilon\|_\infty = 1 \quad \forall \varepsilon \in (0,1)$

$$\Rightarrow \quad \varphi(f_\varepsilon) = \underbrace{\int_{-1}^{-\varepsilon} 1\, dx}_{1-\varepsilon} + \underbrace{\int_{\varepsilon}^{1} 1\, dx}_{1-\varepsilon} + \int_{-\varepsilon}^{\varepsilon} \operatorname{sgn}(x) f_\varepsilon(x)\, dx =$$

$$= 2 - 2\varepsilon + \underbrace{\int_{-\varepsilon}^{\varepsilon} \operatorname{sgn}(x) f_\varepsilon(x)\, dx}_{2\int_0^\varepsilon \frac{x}{\varepsilon} dx = \frac{2}{\varepsilon}\left[\frac{x^2}{2}\right]_0^\varepsilon = \frac{2}{\varepsilon}\cdot\frac{\varepsilon^2}{2} = \varepsilon} = 2 - \varepsilon$$

$\Rightarrow \|\varphi\| \geq 2 - \varepsilon \quad \forall \varepsilon > 0 \quad \Rightarrow$

$$\underline{\underline{\|\varphi\| = 2}}$$

7)

c) $\varphi(u) = \int_{-1}^{1} u(x)\,dx - u(0)$

---

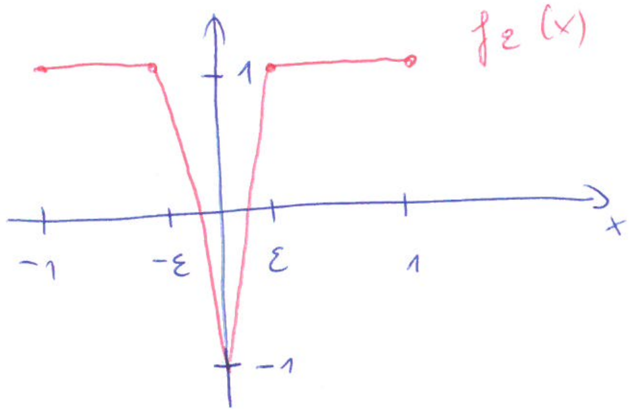
for $u \in C[-1,1]$, $\|u\|_\infty \le 1$

$$|\varphi(u)| = \left| \int_{-1}^{1} u(x)\,dx - u(0) \right| \le \int_{-1}^{1} |u(x)|\,dx + |u(0)| \le$$

$$\le \|u\|_\infty \underbrace{\int_{-1}^{1} dx}_{2} + \underbrace{|u(0)|}_{\substack{\wedge\!\!\setminus \\ \|u\|_\infty}} \le 3\,\|u\|_\infty$$

$$\Rightarrow \quad \|\varphi\| \le 3$$

For any $\varepsilon \in (0,1)$, let define



$$f_\varepsilon(x) = \begin{cases} 1 & \text{if } x \in [-1,-\varepsilon] \\ -\frac{2}{\varepsilon}x - 1 & \text{if } x \in [-\varepsilon, 0] \\ \frac{2}{\varepsilon}x - 1 & \text{if } x \in [0,\varepsilon] \\ 1 & \text{if } x \in [\varepsilon, 1] \end{cases}$$

$$\hookrightarrow \varphi(f_\varepsilon) = \underbrace{\int_{-1}^{-\varepsilon} 1\,dx}_{1-\varepsilon} + \underbrace{\int_{\varepsilon}^{1} 1\,dx}_{1-\varepsilon} + \int_{-\varepsilon}^{\varepsilon} f_\varepsilon(x)\,dx - (-1) = 3 - 2\varepsilon + \underbrace{\int_{-\varepsilon}^{\varepsilon} f_\varepsilon(x)\,dx}_{} \,\stackrel{\circ}{=}$$

$$2 \cdot \int_0^{\varepsilon} \left(\frac{2}{\varepsilon}x - 1\right) dx =$$

$$= 2\left[\frac{2}{\varepsilon}\frac{x^2}{2} - x\right]_0^{\varepsilon} = 0$$

$$= 3 - 2\varepsilon \quad \forall\, \varepsilon > 0$$

$$\Rightarrow \|\varphi\| \ge 3 - 2\varepsilon \quad \Rightarrow \quad \boxed{\|\varphi\| = 3}$$

0!

8/

(9) f positive linear functional on $(C[a,b], \|.\|_\infty)$.

Then $\|f\| = f(\chi_{[a,b]})$.

For any $u \geq 0$, $u \in C[a,b]$:

$$\|u\|_\infty \cdot \chi_{[a,b]} \pm u \geq 0$$

f positive linear
$$\Longrightarrow \quad f(\|u\|_\infty \cdot \chi_{[a,b]} \pm u) = \|u\|_\infty f(\chi_{[a,b]}) \pm f(u) \geq 0$$

$$\hookrightarrow \quad |f(u)| \leq f(\chi_{[a,b]}) \cdot \|u\|_\infty$$

$$\Longrightarrow \quad \|f\| \leq f(\chi_{[a,b]})$$

Since $f(\chi_{[a,b]}) \leq \|f\| \cdot \underbrace{\|\chi_{[a,b]}\|_\infty}_{=1} = \|f\| \quad \rightsquigarrow \|f\| = f(\chi_{[a,b]})$

$\circ$!

(5) Show that $(\mathbb{K}^n, \|.\|_1)^* = (\mathbb{K}^n, \|.\|_\infty)$.

Consider the canonical basis $e_1, \dots, e_n$ in $\mathbb{K}^n$. Then for any

$x = (x_1, \dots, x_n) \in \mathbb{K}^n$, $x = \sum_{i=1}^n x_i e_i$ and

$$\underset{\substack{\uparrow \\ \varphi \in B(\mathbb{K}^n, \mathbb{K})}}{\varphi(x) = \varphi\left(\sum_{k=1}^n x_k e_k\right)} = \sum_{k=1}^n x_k \varphi(e_k) \underset{\substack{\uparrow \\ \varphi(e_k) := \alpha_k}}{=} \sum_{k=1}^n x_k \alpha_k$$

$\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{K}^n$

9)

$$\Rightarrow \quad |\varphi(x)| = \left|\sum_{k=1}^{n} x_k \alpha_k\right| \leq \sum_{k=1}^{n} |x_k \alpha_k| \leq \|\alpha\|_\infty \sum_{k=1}^{n} |x_k| =$$

$$= \|\alpha\|_\infty \cdot \|x\|_1$$

$$\Longrightarrow \quad \|\varphi\| \leq \|\alpha\|_\infty .$$

There exists an index $\ell \in \{1, .., n\}$ s.t.

$\|\alpha\|_\infty = |\alpha_\ell|$, hence

$$\|\alpha\|_\infty = |\varphi(e_\ell)| \leq \sup\{|\varphi(x)| \in \mathbb{R} : \|x\|_1 \leq 1\} = \|\varphi\|$$

$$\Longrightarrow \quad \|\varphi\| = \|\alpha\|_\infty .$$

So $\quad \Phi : (\mathbb{K}^n, \|.\|_\infty) \longrightarrow (\mathbb{K}^n, \|.\|_1)^*$

$$\alpha = (\alpha_1, .., \alpha_n) \longmapsto \varphi_\alpha$$

$\varphi_\alpha(x) := \sum_{k=1}^{n} \alpha_k x_k$ is an isometric isomorphism

ie $\quad (\mathbb{K}^n, \|.\|_1)^* = (\mathbb{K}^n, \|.\|_\infty)$ $\quad$ $\circ$!

10/ ⑥ $x=(x_n)_{n\in\mathbb{N}} \in C$ with $\|\cdot\|_\infty$

$(e_n)_{n\in\mathbb{N}}$: standard basis , $\alpha(x) := \lim_n x_n$

$e := (1,1,\dots)$. Then for any $\varphi \in C^*$

a) $(\varphi(e_n))_{n\in\mathbb{N}} \in \ell_1$

Since for any $\varphi \in C^* \Rightarrow \varphi|_{C_0} \in C_0^* \Rightarrow (\varphi(e_n)) \in \ell_1$

(we proved it in the lecture) $\circ$!

b) $\varphi(x) = \alpha(x)\left(\varphi(e) - \sum_{n=1}^{\infty}\varphi(e_n)\right) + \sum_{n=1}^{\infty} x_n\varphi(e_n)$

For the sequence

$$y_n := \alpha(x)e + \sum_{k=1}^{n}(x_k - \alpha(x))e_k \qquad n\in\mathbb{N}$$

$x=(x_n)\in C$

$$\lim_{n\to\infty} y_n = \alpha(x)e + \underbrace{\sum_{k=1}^{\infty}(x_k-\alpha(x))e_k}_{\underbrace{\sum_{k=1}^{\infty}x_k e_k}_{x} - \alpha(x)\underbrace{\sum_{k=1}^{\infty}e_k}_{e}} = x$$

$$\Rightarrow \varphi(x) = \varphi(\lim_{n\to\infty} y_n) \underset{\substack{\uparrow \\ \varphi\in C^* \Rightarrow \varphi \text{ is continuous}}}{=} \lim_{n\to\infty}\varphi(y_n) \underset{\substack{\uparrow \\ \varphi \text{ is linear}}}{=}$$

$$= \lim_{n\to\infty}\left[\alpha(x)\varphi(e) + \sum_{k=1}^{n}(x_k-\alpha(x))\varphi(e_k)\right] =$$

$$= \alpha(x)\varphi(e) + \sum_{k=1}^{\infty}x_k\varphi(e_k) - \alpha(x)\sum_{k=1}^{\infty}\varphi(e_k)$$

$\circ$!

11)

c) $\|\varphi\| = |\varphi(e) - \sum_{n=1}^{\infty} \varphi(e_n)| + \sum_{n=1}^{\infty} |\varphi(e_n)|$

By b) for any $x = (x_n)_{n\in\mathbb{N}} \in C$

$$|\varphi(x)| \leq \underbrace{|\alpha(x)|}_{\leq \|x\|_\infty} \cdot \underbrace{|\varphi(e) - \sum_{n=1}^{\infty} \varphi(e_n)|}_{=: \chi_\varphi} + \sum_{n=1}^{\infty} |x_n|\,|\varphi(e_n)| \leq$$

$$\leq \left(|\chi_\varphi| + \sum_{n=1}^{\infty} |\varphi(e_n)| \right) \cdot \|x\|_\infty$$

$$\Rightarrow \quad \|\varphi\| \leq |\chi_\varphi| + \sum_{n=1}^{\infty} |\varphi(e_n)|$$

For any $n \in \mathbb{N}$

$$u_n := \operatorname{sgn}(\varphi(e_n)) \quad , \quad v_n := \operatorname{sgn}\left(\varphi(e) - \sum_{k=1}^{n} \varphi(e_k)\right)$$

$$w_k^{(n)} := \begin{cases} u_k & \text{if } k \leq n \\ v_n & \text{if } k > n \end{cases} .$$

Then for the sequence $w^{(n)} := (w_k^{(n)})_{k\in\mathbb{N}}$: $w^{(n)} \in C$

and $\lim_{k\to\infty} w_k^{(n)} = v_n \quad \forall n \in \mathbb{N}$, $\|w^{(n)}\|_\infty \leq 1$

$$\Rightarrow \quad |\varphi(e) - \sum_{k=1}^{n} \varphi(e_k)| + \sum_{k=1}^{n} |\varphi(e_k)| = \left| v_n\left(\varphi(e) - \sum_{k=1}^{n} \varphi(e_k)\right) + \sum_{k=1}^{n} u_k \varphi(e_k) \right|$$

$$= \left| v_n \chi_\varphi + \sum_{k=1}^{n} u_k \varphi(e_k) + \sum_{k=n+1}^{\infty} v_n \varphi(e_k) \right| =$$

$$= \left| v_n \chi_\varphi + \sum_{k=1}^{\infty} w_k^{(n)} \varphi(e_k) \right| = |\varphi(w^{(n)})| \leq \|\varphi\| \cdot \|w^{(n)}\|_\infty \leq \|\varphi\|$$

with $n \to \infty$ we get $|\chi_\varphi| + \sum_{k=1}^{\infty} |\varphi(e_k)| \leq \|\varphi\|$ $\quad$. !

⑦ Prove that $c^* = \ell_1$

With the notation of ⑥, consider the map

$$\Phi : (c, \|\cdot\|_\infty)^* \longrightarrow (\ell_1, \|\cdot\|_1)$$

$$\varphi \longmapsto (u_n)_{n\in\mathbb{N}}$$

where $u_1 := \chi_\varphi$, $u_{n+1} := \varphi(e_n)$ $\forall n \in \mathbb{N}$.

For any $\varphi \in c^*$:

$$\|\Phi(\varphi)\|_1 = \sum_{n=1}^{\infty} |u_n| = |\chi_\varphi| + \sum_{n=1}^{\infty} |\varphi(e_n)| = \|\varphi\| .$$

For any $u = (u_n)_{n\in\mathbb{N}} \in \ell_1$ define the functional

$$\varphi_u : c \longrightarrow \mathbb{K}$$

$$(x_n) \longmapsto u_1 \lim_n x_n + \sum_{n=1}^{\infty} x_n u_{n+1}$$

↳ φ is linear and

$$|\varphi_u(x)| \leq \underbrace{|u_1|}_{\leq \|u\|_1} \cdot \|x\|_\infty + \|u\|_1 \cdot \|x\|_\infty \leq 2\|u\|_1 \cdot \|x\|_\infty$$

$\Rightarrow$ φ_u is bounded i.e. $\varphi_u \in c^*$

Since $\varphi_u(e_n) = u_{n+1}$ and

$$\chi_{\varphi_u} = \varphi_u(e) - \sum_{n=1}^{\infty} \varphi_u(e_n) = u_1 + \sum_{n=1}^{\infty} u_{n+1} - \sum_{n=1}^{\infty} u_{n+1} = u_1$$

we have $\Phi(\varphi_u) = u$

$\Rightarrow$ Φ is isometrical isomorphism !

(8) Let X be a normed space. Show that if for some $x \in X$

$\varphi(x) = 0$ holds for all $\varphi \in X^*$, then $x = 0$.

We have seen in the lecture, that for any $0 \neq x \in X$

there exists $\varphi \in X^*$ s.t. $\|\varphi\| = 1$ and $\varphi(x) = \|x\|$.

So if $x \neq 0$, there exists $\varphi \in X^*$ s.t. $\varphi(x) = \|x\| \neq 0$

which is contradiction $\Rightarrow$ $x = 0$!

(9) Let X be a normed space. Show that if for any $x, y \in X$

~~& $x \neq y$ there exists $\varphi \in X^*$, s.t. $\|\varphi\| = 1$ and $\varphi(x) \neq \varphi(y)$.~~

$\varphi(x) = \varphi(y)$ $\forall \varphi \in X^* \Rightarrow x = y$

If $\varphi(x) = \varphi(y)$ for all $\varphi \in X^*$, then by the linearity of φ

$$\varphi(x - y) = 0 \text{ for all } \varphi \in X^* \xRightarrow{(8)} x - y = 0$$

i.e. $x = y$!

(10) Let X be a normed space. Show that for any $x, y \in X$, $x \neq y$

$\exists$ $\varphi \in X^*$ s.t. $\|\varphi\| = 1$ and $\varphi(x) \neq \varphi(y)$.

We have seen: $\forall x \neq 0$ $\exists$ $\varphi \in X^*$ s.t. $\|\varphi\| = 1$ and $\varphi(x) = \|x\|$

$\Rightarrow$ if $x \neq y$ $\Rightarrow$ $u = x - y \neq 0$ $\Rightarrow$ $\exists$ $\varphi \in X^*$ s.t.

$$\|\varphi\| = 1 \text{ and } \varphi(u) = \varphi(x - y) = \varphi(x) - \varphi(y) = \|u\| \neq 0$$

$$\Downarrow$$

$$\varphi(x) \neq \varphi(y)$$

!

12 14

11

(11) X normed space, $f \in X'$

$f \in X^* \iff \ker f$ is closed linear subspace in X

If $f \in X^*$ then f is continuous and $\ker f = f^{-1}(\{0\})$ is closed because the preimage of the closed set $\{0\}$.

Suppose that $f \not\equiv 0$. Then there exists $x \in X$ st $f(x) = 1$.

Suppose that $\ker f$ is closed, but f is not continuous.

Then there exists a sequence $(z_n)_{n\in\mathbb{N}} \subset X$, $\|z_n\| = 1$ $\forall n$ st

$|f(z_n)| \longrightarrow \infty$, as f is not bounded.

$$\frac{z_n}{f(z_n)} - x \in \ker f$$

but $$\frac{z_n}{f(z_n)} - x \xrightarrow[n\to\infty]{} -x$$ and $-x \in \ker f$ since $\ker f$ is closed

$\Downarrow$

$\lightning$ because $f(x) = 1$

!