Functional Analysis,
Exercises 4.

1. Let $(X, \|.\|_X)$ be a Banach space and $(Y, \|.\|_Y)$ be a normed space. Show that if the operator sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{B}(X, Y)$ is pointwisely convergent, then the (pointwisely) limit operator given by

$$Ax := \lim_{n\to\infty} A_n x, \quad (x \in X)$$

is bounded, i.e. $A \in \mathcal{B}(X, Y)$ and

$$\|A\| \leq \liminf_n (\|A_n\|) \leq \sup\{\|A_n\| : n \in \mathbb{N}\} < \infty.$$

2. Let $(X, \|.\|_X)$ be a Banach space and $(Y, \|.\|_Y)$ be a normed space and let $\mathcal{F} \subset \mathcal{B}(X, Y)$ be a nonempty subset. Show that if $\mathcal{F}$ is not uniformly bounded, i.e.

$$\sup\{\|A\| : A \in \mathcal{F}\} = \infty,$$

then $\mathcal{F}$ is not pointwisely bounded, i.e. there exist $u \in X$ s.t.

$$\sup\{\|Au\|_Y : A \in \mathcal{F}\} = \infty.$$

Moreover, show that the set

$$\mathcal{A} := \{u \in X : \sup\{\|Au\|_Y : A \in \mathcal{F}\} = \infty\}$$

is dense in X.

3. Let $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ be normed spaces and $A \in \mathcal{B}(X, Y)$. Show that A is an open map iff there exist $r > 0$ s.t

$$\mathcal{B}_Y(0, r) \subset A\left(\mathcal{B}_X(0, 1)\right).$$

($\mathcal{B}_Z(z_0, r)$ denotes the open ball centred at z_0 with radius r in the normed space Z.)

4. Show that the linear operator defined by

$$A : \ell_\infty \to \mathbf{c}_0, \quad A(x_n)_{n\in\mathbb{N}} := \left(\frac{x_n}{n}\right)_{n\in\mathbb{N}}$$

is not an open map.

5. Let $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ be normed spaces and $A : X \to Y$. Prove that A is closed operator iff for any sequence $(x_n)_{n\in\mathbb{N}} \subset \mathcal{D}(A)$ with $x_n \to x$ and $Ax_n \to y \in Y$, $x \in \mathcal{D}(A)$ and $y = Ax$ hold.

6. Let $X = Y = C[0, 1]$ with $\|.\|_\infty$ norm and consider the operator

$$A : C^1[0, 1] \to Y, \quad Af := f'.$$

Show that $A : X \to Y$ is a closed operator. ($\mathcal{D}(A) = C^1[0, 1] \subset X$)

7. Let $X = Y = L^2[-1, 1]$ and consider the operator

$$A : C^1[-1, 1] \to Y, \quad Af := f'.$$

Show that $A : X \to Y$ is not a closed operator. ($\mathcal{D}(A) = C^1[-1, 1] \subset X$)

8. Let $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ be normed spaces and $A : X \to Y$ a linear operator. Prove the following statements.

   (a) If A is closed operator then $\mathrm{Ker}(A)$ is a closed subspace in X.

   (b) If A is injective, then A is closed iff A^{-1} is closed.

   (c) If $A \in \mathcal{B}(X, Y)$ then A is closed.

   (d) If $(Y, \|.\|_Y)$ is Banach space, A is bounded and closed, then $\mathcal{D}(A)$ is closed.

   (e) If $(X, \|.\|_X)$ is Banach space, A is bounded, closed and injective and A^{-1} is bounded, then $\mathrm{Ran}(A)$ is closed.

9. Let $(X, \|.\|)$ be a Banach space and $A : X \to X$ a projection, i.e. A is a linear operator, s.t. $A^2 = A$. Show that $A \in \mathcal{B}(X)$ iff $\mathrm{Ker}(A)$ and $\mathrm{Ran}(A)$ are closed.

10. Let $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ be normed spaces and $A : X \to Y$ linear. Show that

$$\|x\|_A := \sqrt{\|x\|_X^2 + \|Ax\|_Y^2}, \quad (x \in \mathcal{D}(A))$$

defines a norm. Prove that if $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ are Banach spaces, the A is closed iff $(\mathcal{D}(A), \|.\|_A)$ is Banach space.