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Functional Analysis, Exercises 4
Solutions

(1) $(X, \|.\|_X)$ Banach space, $(Y, \|.\|_Y)$ normed space

If $(A_n)_{n\in\mathbb{N}} \subset B(X,Y)$ is pointwisely convergent, then

$$Ax := \lim_{n\to\infty} A_n x \quad x \in X \quad \text{defines a bounded operator}$$

and

$$\|A\| \le \liminf_n (\|A_n\|) \le \sup\{\|A_n\| : n\in\mathbb{N}\} < \infty.$$

$(A_n)_{n\in\mathbb{N}}$ is pointwisely convergent $\Rightarrow$ $(A_n)_{n\in\mathbb{N}}$ is pointwisely Cauchy

i.e. $\|A_n x - A_m y\|_Y \to 0$ if $m,n\to\infty$

$+$ $(X, \|.\|)_X$ is complete

$\Downarrow$ Uniform Boundedness Thm

$(A_n)_{n\in\mathbb{N}}$ is uniformly bounded

$\hookrightarrow$ $$\|Ax\|_Y = \lim_{n\to\infty} (\|A_n x\|_Y) \le \sup\{\|A_n\| : n\in\mathbb{N}\}\cdot\|x\|_X$$

$\uparrow$ the norm is continuous $\Rightarrow$ A is bounded ✓

As $\liminf(\|A_n\|) \le \sup\{\|A_n\| : n\in\mathbb{N}\}$ is always true,

we can assume that $\alpha := \liminf(\|A_n\|) < \infty$.

$\Rightarrow$ $\exists$ $(n_k)_{k\in\mathbb{N}}$ subsequence of indices s.t.

$$\alpha = \lim_k (\|A_{n_k}\|)$$

$\Rightarrow$ $\forall x\in X$

$$\|Ax\|_Y = \lim_{n\to\infty}(\|A_n x\|_Y) = \lim_{k\to\infty}(\|A_{n_k} x\|_Y) \le \lim_{k\to\infty}(\|A_{n_k}\|)\cdot\|x\|_X =$$

$$= \alpha\cdot\|x\|_X \Rightarrow \|A\| \le \alpha$$

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(2) $(X, \|.\|_X)$ Banach space, $(Y, \|.\|_Y)$ normed space

$\mathcal{F} \subset B(X,Y)$ non-empty. If $\mathcal{F}$ is not uniformly bounded

i.e. $\sup\{\|A\| : A \in \mathcal{F}\} = \infty \Rightarrow \mathcal{F}$ is not pointwisely bounded
i.e. $\exists\, u \in X$: $\sup\{\|Au\|_Y : A \in \mathcal{F}\} = \infty$

and $\mathcal{A} := \{u \in X : \sup\{\|Au\|_Y : A \in \mathcal{F}\} = \infty\}$ is dense in X.

Uniform Boundedness Thm: $\mathcal{F}$ is pointwisely continuous $\Rightarrow$ $\mathcal{F}$ is uniformly continuous

$\hookrightarrow$ $\mathcal{F}$ is not pointwisely continuous, because $\mathcal{F}$ is not uniformly continuous

If $\overline{\mathcal{A}} \neq X$, then $\exists\, x \in X$ and a neighborhood U of X s.t.

$$U \cap \mathcal{A} = \emptyset \quad \text{i.e.}$$

$$\sup\{\|Ax\|_Y : A \in \mathcal{F}\} < \infty \quad x \in U$$

$$\Downarrow$$

$\mathcal{F}$ is pointwisely bounded : $\lightning$

(3) $(X, \|.\|_X), (Y, \|.\|_Y)$ normed spaces, $A \in B(X,Y)$.

A is an open map $\Longleftrightarrow \exists\, r > 0 : B_Y(0,r) \subset A(B_X(0,1))$

$\Rightarrow$: If A is open, then for all $U \subset X$ open $A(U) \subset Y$ is open.

$\hookrightarrow$ $B_X(0,1) \subset X$ is open in X $\Rightarrow A(B_X(0,1))$ is open in Y

A is linear $\Rightarrow$ for $0 \in X$ $A0 = 0 \in Y$ and $0 \in A(B_X(0,1))$

and $0 \in \operatorname{int} A(B_X(0,1))$

$$\Downarrow$$

$$\exists\, r > 0 \quad B_Y(0,r) \subset A(B_X(0,1))$$

3/

$\Leftarrow$: If $\emptyset \neq \mathcal{U} \subset X$ is open and $y \in A(\mathcal{U})$

$\Downarrow$

y is an inner point of $A(\mathcal{U})$

$\Downarrow$

$\exists\, \delta > 0$ st $B_Y(y, \delta) \subset A(\mathcal{U})$.

Since $y \in A(\mathcal{U}) \Rightarrow \exists\, x \in X : y = Ax$.

$\mathcal{U}$ is open $\Rightarrow \exists\, \varepsilon > 0\ B_X(x, \varepsilon) \subset \mathcal{U}$. $(*)$

Assumption: $B_Y(0, r) \subset A(B_X(0,1))$ + A is linear

$$\Rightarrow \quad B_Y(0, \varepsilon \cdot r) \subset A(B_X(0, \varepsilon))$$

And

$$Ax + B_Y(0, \varepsilon \cdot r) = y + B_Y(0, \varepsilon \cdot r) = \{y + y' \in Y : y' \in Y : \|y'\|_Y < \varepsilon r\}$$

$$= \{b \in Y : \|b - y\|_Y < \varepsilon r\} = B_Y(y, \varepsilon r) \subset$$

$$\subset Ax + A(B_X(0, \varepsilon)) = A(x + B_X(0, \varepsilon)) =$$

$$= A(B_X(x, \varepsilon)) \underset{(*)}{\subset} A(\mathcal{U})$$

$\Rightarrow A$ is open

.!

(5) $A : \ell_\infty \to c_0$, $A(x_n)_{n\in\mathbb{N}} := \left(\frac{x_n}{n}\right)_{n\in\mathbb{N}}$ is not an open map.

$$A(B(0,1)) = \{(x_n)_{n\in\mathbb{N}} \subset c_0 : |x_n| < \tfrac{1}{n}\ n \in \mathbb{N}\}$$

$\|$

$\{(x_n)_{n\in\mathbb{N}} \subset \ell_\infty : \sup |x_n| < 1\}$

$\Downarrow$

$\nexists\, r > 0$ st

$B(0, r) = \{(x_n)_{n\in\mathbb{N}} \subset \ell_\infty : |x_n| < r\} \subset$

$\subset A(B(0,1))$

and by the exercise (3) $\Rightarrow$ A is not open. .!

4)

(5) $(X,\|.\|_X),(Y,\|.\|_Y)$ normed spaces, $A: X\to Y$.

A is closed operator iff $\forall (x_n)_{n\in\mathbb{N}}\subset D(A)$, $x_n\to x$ and $Ax_n\to y$, $x\in D(A)$ and $y=Ax$ hold.

$\Gamma(A)$ is closed iff $\forall (u,v)\in\overline{\Gamma(A)} \Rightarrow (u,v)\in\Gamma(A)$.

$(u,v)\in\overline{\Gamma(A)} \iff \exists (u_n,v_n)_{n\in\mathbb{N}}\subset\Gamma(A)$ st $\lim\limits_{n\to\infty}(u_n,v_n)=(u,v)$

Γ in the norm $\|(.,.)\|_{X\times Y}$

ie $\|u_n-u\|_X\to 0$ and $\|Au_n-v\|=\|v_n-v\|\to 0$ as $n\to\infty$

$\Rightarrow$ $\Gamma(A)$ is closed if $\forall (u_n)_{n\in\mathbb{N}}\subset D(A)$ st $u_n\xrightarrow{\|.\|_X}u$ and $Au_n\xrightarrow{\|.\|_Y}v$ $\Rightarrow u\in D(A)$ and $v=Au$

.!

(6) $X=Y=C[0,1]$, $A: C^1[0,1]\to Y$, $Af:=f'$.

$\Rightarrow$ $A: X\to Y$ is closed. $(D(A)=C^1[0,1]\subset X)$.

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let $(f_n)_{n\in\mathbb{N}}\subset C^1[0,1]$, $\lim\limits_{n\to\infty} f_n=f$ and $\lim\limits_{n\to\infty}(Af_n)=g$

$\hookrightarrow$ $\|Af_n-g\|_Y=\|f_n'-g\|_Y\xrightarrow[n\to\infty]{}0$ ie $(Af_n)_{n\in\mathbb{N}}\equiv(f_n')_{n\in\mathbb{N}}$ is uniformly convergent

$\Rightarrow$ $\forall x\in[0,1]$ $\displaystyle\int_0^x f_n'(t)\,dt \longrightarrow \int_0^x g(t)\,dt$ $(n\to\infty)$

ie $f_n(x)-f_n(0)=\displaystyle\int_0^x f_n'(t)\,dt\to\int_0^x g(t)\,dt$

since $\lim\limits_{n\to\infty} f_n=f$ $\Rightarrow$ $f(x)-f(0)=\displaystyle\int_0^x g(t)\,dt$ $\Rightarrow$ $f\in C^1[0,1]$ and $Af=f'=g$.!

5)

(7) $X = Y = L^2[-1,1]$

$A : C^1[-1,1] \to Y \quad , Af := f'$

$\Rightarrow \quad A : X \to Y$ is not closed. $\quad (D(A) = C^1[-1,1] \subset X)$

For the sequence $\qquad f_n(x) := \sqrt{x^2 + \frac{1}{n}} \qquad x \in [-1,1] \; , n \in \mathbb{N}$

$$\lim_{n\to\infty} f_n(x) = \sqrt{x^2} = |x| =: f(x) \qquad x \in [-1,1]$$

~~$Af = f'$~~

$$\cancel{(Af)(x) = f'(x)} \; g(x) := \begin{cases} -1 & \text{if } x<0 \\ 0 & \text{if } x=0 \\ 1 & \text{if } x>0 \end{cases}$$

$$\lim_{n\to\infty} \|f_n - f\|_2 = \lim_{n\to\infty} \sqrt{\int_{-1}^{1} (f_n(x) - f(x))^2 \, dx} = \sqrt{\int_{-1}^{1} \lim_{n\to\infty} (f_n(x) - f(x))^2 \, dx} = 0$$

ie $\quad f_n \xrightarrow{L^2} f$

and

~~but~~ $$\lim_{n\to\infty} \|f_n' - g\|_2 = \lim_{n\to\infty} \sqrt{\int_{-1}^{1} (f_n'(x) - g(x))^2 \, dx} =$$

$$= \sqrt{\int_{-1}^{1} \lim_{n\to\infty} (f_n'(x) - g(x))^2 \, dx} = 0$$

$$f_n'(x) = \frac{x}{\sqrt{x^2 + \frac{1}{n}}} \xrightarrow[n\to\infty]{} g(x)$$

ie $\quad Af_n \xrightarrow{L_2} g$

but $$(Af)(x) = f'(x) = \begin{cases} -1 & \text{if } x<0 \\ 1 & \text{if } x>0 \end{cases} \neq g(x)$$

ie $\quad Af \neq g \Rightarrow A$ is not closed $\circ$!

6/ (8) $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ normed space, $A : X \to Y$ linear op

a) A is closed $\Rightarrow$ Ker A is closed subspace in X

$(x_n)_{n\in\mathbb{N}} \subset \operatorname{Ker} A \subset D(A)$

A is closed $\Rightarrow$ $\left.\begin{array}{l} x := \lim\limits_{n\to\infty} x_n \in X \\ \lim\limits_{n\to\infty} A x_n = 0 \in Y \end{array}\right\} \Rightarrow x \in D(A)$ and $0 = Ax$

$\Downarrow$

$x \in \operatorname{Ker} A$

$\Downarrow$

Ker A is closed .!

b) A is injective, then A is closed iff A^{-1} is closed

Let A be injective and closed, $(y_n)_{n\in\mathbb{N}} \in \operatorname{Ran} A = \cancel{\operatorname{Ker}} D(A^{-1})$

s.t. $\lim\limits_{n\to\infty} y_n =: y \in Y$ and $\lim (A^{-1} y_n) =: x \in X$,

then $x \in D(A)$ and $Ax = y$, so $y \in D(A^{-1})$ and $A^{-1} y = x$

$\Rightarrow$ A^{-1} is closed.

If A^{-1} is closed, then $A = (A^{-1})^{-1}$ is also closed .!

c) $A \in \mathcal{B}(X, Y) \Rightarrow$ A is closed.

If $(x_n)_{n\in\mathbb{N}} \subset X$ s.t. $x = \lim\limits_{n\to\infty} x_n \in X$ and $\lim (A x_n) =: y \in Y$

then $x \in X$ and

$$\underbrace{\lim (A x_n) = A x}_{A \text{ is continuous}} \Rightarrow Ax = y$$

$\Downarrow$

A is closed.

7/ d) $(Y, \|.\|_Y)$ is Banach, A is bounded and closed $\Rightarrow$ $D(A)$ is closed.

For $(x_n)_{n\in\mathbb{N}} \subset D(A)$ s.t. $\lim_{n\to\infty} x_n = x \in X$ $\Rightarrow$ $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence

$\Downarrow$ A is bounded

$(Ax_n)_{n\in\mathbb{N}}$ is Cauchy sequence

$\Downarrow$ Y is Banach

$\exists\, y = \lim_{n\to\infty}(Ax_n)$

A is closed

$\Downarrow$

$x \in D(A)$ and $y = Ax$

$\hookrightarrow$ $D(A)$ is closed.

e) $(X, \|.\|_X)$ is Banach, A is bounded, closed and injective $\Rightarrow$ A^{-1} is closed and $\operatorname{Ran} A$ is closed

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By b) $\Rightarrow$ A^{-1} is closed $\overset{d)}{\Rightarrow}$ $D(A^{-1}) = \operatorname{Ran}(A)$ is closed. !

(9) $(X, \|.\|)$ Banach space, $A: X \to X$ is a projection (linear, ~~bounded~~ and $A^2 = A$)

A is bounded $\Leftrightarrow$ $\operatorname{Ker} A$ and $\operatorname{Ran} A$ are closed.

$\Rightarrow$: A is bounded $\Rightarrow$ $(I-A)^2 = I - 2A + A^2 = I - A$

$\hookrightarrow$ $I-A$ is bounded projection

$\hookrightarrow$ $\boxed{\operatorname{Ker} A = \operatorname{Ran}(I-A)}$

indeed: if $x \in \operatorname{Ker} A$ $\quad Ax = 0$ $\Rightarrow$ $(I-A)x = x$ $\rightsquigarrow$ $x \in \operatorname{Ran}(I-A)$

if $y \in \operatorname{Ran}(I-A)$: $\exists x$: $(I-A)(x) = y \Leftrightarrow x - Ax = y$

$A(y) = Ax - A^2x = 0 \Rightarrow y \in \operatorname{Ker} A$ —

8)

similarly:
$$\operatorname{Ran} A = \operatorname{Ker}(I-A)$$

$I-A$ is bounded $\Rightarrow$ $\operatorname{Ran}(I-A)$ and $\operatorname{Ker}(I-A)$ are closed

$\Downarrow$

$\operatorname{Ker} A$, $\operatorname{Ran} A$ are closed.

$\Leftarrow$: ! $u_n \in X$ $n \in \mathbb{N}$ s.t. $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty}(Au_n) = v$

$\Rightarrow$ $Au_n \in \operatorname{Ran} A$ $n \in \mathbb{N}$.

Since $\operatorname{Ran} A$ is closed $\Rightarrow$ $v \in \operatorname{Ran} A$

$\forall n \in \mathbb{N}$: $$A(Au_n - u_n) = A^2u_n - Au_n = Au_n - Au_n = 0$$

$\Leftrightarrow$ $Au_n - u_n \in \operatorname{Ker} A$ $n \in \mathbb{N}$

$+$ $\operatorname{Ker} A$ is closed $\Rightarrow$ $\lim_{n\to\infty}(Au_n - u_n) = v - u \in \operatorname{Ker} A$

$\Rightarrow$ $u = (u-v)+v$ $\Rightarrow$ $Au = v$

$\Downarrow$

$\Gamma(A)$ is closed

$\Downarrow$

A is bounded $\circ$!

9/

(10) $(X, \|.\|_X)$, $(Y, \|.\|_Y)$ normed spaces. $A: X \to Y$ is linear.

$$\|x\|_A := \sqrt{\|x\|_X^2 + \|Ax\|_Y^2} \quad x \in D(A) \text{ is a norm.}$$

If $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ are Banach spaces, then

$$A \text{ is closed} \iff (D(A), \|.\|_A) \text{ is Banach space.}$$

$\|.\|_A : D(A) \to \mathbb{R}$

- $x \in D(A)$ with $0 = \|x\|_A = \sqrt{\|x\|_X^2 + \|Ax\|_Y^2} \implies \|x\|_X^2 = 0$

$$\Downarrow$$
$$x = 0 \in D(A)$$

- $\lambda \in \mathbb{K}$; $x \in D(A)$

$$\|\lambda x\|_A = \sqrt{\|\lambda x\|_X^2 + \|A(\lambda x)\|_Y^2} = \sqrt{|\lambda|^2 \|x\|_X^2 + |\lambda|^2 \|Ax\|_Y^2} = |\lambda| \cdot \|x\|_A$$

- $x, y \in D(A)$

$$\|x+y\|_A = \sqrt{\underbrace{\|x+y\|_X^2}_{\leq (\|x\|_X + \|y\|_Y)^2} + \underbrace{\|A(x+y)\|_Y^2}_{\leq (\|Ax\|_Y + \|Ay\|_Y)^2}} \leq \sqrt{\|x\|_X^2 + \|Ax\|_Y^2} + \sqrt{\|y\|_X^2 + \|Ay\|_Y^2} =$$

$$= \|x\|_A + \|y\|_A$$

one can check by squaring that

$$\sqrt{(a+b)^2 + (c+d)^2} \leq \sqrt{a^2+c^2} + \sqrt{b^2+d^2}$$

10)

Let $(x_n)_{n\in\mathbb{N}} \subset (D(A), \|.\|_A)$ be a Cauchy sequence

$$\Downarrow$$

$(x_n)_{n\in\mathbb{N}}$ is Cauchy in $(X, \|.\|_X)$

$(Ax_n)_{n\in\mathbb{N}}$ –||– in $(Y, \|.\|_Y)$

$\Downarrow$ X and Y are complete

$\exists\, x \in X, y \in Y$ s.t.

$$\lim_{n\to\infty} \|x_n - x\|_X = 0 \quad \text{and} \quad \lim_{n\to\infty} \|Ax_n - Ax\|_Y = 0$$

A is closed $\Rightarrow x \in D(A)$ and $Ax = y$

$$\hookrightarrow \|x - x_n\|_A = \sqrt{\|x - x_n\|_X^2 + \|A(x - x_n)\|_Y^2} \to 0 \quad (n \to \infty)$$

$$\Downarrow$$

$(x_n)_{n\in\mathbb{N}}$ is convergent in $(D(A), \|.\|_A)$

$$\Downarrow$$

$D(A)$ is a Banach space.

11) $(D(A), \|.\|_A)$ is a Banach space, $(x_n)_{n\in\mathbb{N}} \subset D(A)$ s.t.

$$\lim_{n\to\infty} \|x_n - x\|_X = 0 \quad \text{and} \quad \lim_{n\to\infty} \|Ax_n - y\|_Y = 0$$

$\Rightarrow (x_n)_{n\in\mathbb{N}}$ is Cauchy in $(X, \|.\|_X)$ and $(Ax_n)_{n\in\mathbb{N}}$ is Cauchy in $(Y, \|.\|_Y)$.

$$\Downarrow$$

it is Cauchy in $(D(A), \|.\|_A)$ too.

$$\Downarrow$$

$$\exists\, u \in D(A) : \lim_{n\to\infty} \|x_n - u\|_A = 0$$

$\Rightarrow \lim_{n\to\infty} \|x_n - u\|_X = 0$, $\lim \|Ax_n - Au\|_Y = 0 \Rightarrow x = u \in D(A)$ and $Au = Ax = y$

A is closed.