Functional Analysis,
Exercises 5.

Recall that every normed space X can be isometrically embedded into its bidula by the map $(Jx)(\varphi) := \varphi(x)$, $x \in X$, $\varphi \in X^*$. X is called reflexive if J is a bijection.

1. Show that ℓ_p is reflexive for every $p \in (1, \infty)$.

2. Let X be a normed space. Prove that

   (a) If X is finite dimensional then it is reflexive,

   (b) If X is reflexive and separable then X^* is separable.

   (c) If X is reflexive then X^* is reflexive.

   (d) If X is a Banach space and X^* is reflexive then X is reflexive.

3. Let X, Y be normed spaces and $T : X \to Y$ a linear operator. Show that the following are equivalent.

   (a) The graph $\Gamma(T)$ of T is closed.

   (b) If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X s.t. $x_n \to 0$ and $(Tx_n)_{n\in\mathbb{N}}$ converges, then $\lim_n Tx_n = 0$.

4. By definition, the weak topology $\sigma(X, X^*$ on a normed space X is the weakest topology on X s.t. all elements in X^* are continous w.r.t. it. Show that the weak toipology is generated by the sets
$$U(c, f, \varepsilon) := f^{-1}(\mathcal{B}(c, \varepsilon)) = \{x \in X : |f(x) - c| < \varepsilon\}, \quad c \in \mathbb{K}, f \in X^*, \varepsilon > 0,$$
   where $\mathcal{B}(c, \varepsilon) = \{x \in \mathbb{K} : |x - c| < \varepsilon\}$.

5. Let X be an infinite dimensional normed space and $S_X := \{x \in X : \|x\| = 1\}$ be the unit sphere of X. Show the followin surprising fact: the closure of the unit sphere in the weak topology is the whole closed unit ball, i.e.
$$\overline{\{x \in X : \|x\| = 1\}}^{\sigma(X,X^*)} = \{x \in X : \|x\| = 1\}.$$
   Conclude that the weak topology and the norm topology are different on any infnite dimensional normed space.

6. Let $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ be normed spaces and $T : X \to Y$ a linear operator. Prove that the following statements are equivalent.

   (a) T is continous.

   (b) For every sequence $(x_n)_{n\in\mathbb{N}}$ in X, weak convergence $x_n \to^w x$ in X implies weak convergence $Tx_n \to^w Tx$ in Y.

7. Let $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ be Banach spaces and let $T : \mathcal{D}(T) \subset X \to Y$ be a linear operator with closed graph. Prove that the following statements are equivalent.

   (a) T is injective and its range $\mathrm{Ran}(T) = T(\mathcal{D}(T))$ is closed in Y

   (b) There exists $C > 0$ s.t. for all $x \in \mathcal{D}(T)$ $\|x\|_X \leq C\|Tx\|_Y$.