Compact operators

Recall: A metric space $\mathcal{M}$ is

- compact iff every open cover of $\mathcal{M}$ contains a finite subcover
- totally bounded iff for every $\varepsilon > 0$ there exists a covering of $\mathcal{M}$ with finitely many open balls of radius ε.
- $E \subset \mathcal{M}$ is precompact iff $\bar{E}$ is compact.

FACTS
- If $\mathcal{M}$ is complete, then $E \subset \mathcal{M}$ is precompact iff it is totally bounded
- If X is finite dimensional normed space, then every bounded set is precompact.
- The unit ball in an infinite-dimensional space is not precompact.

Def: Let X, Y be Banach spaces. A linear operator $A: X \to Y$ is called a compact operator if it maps the unit ball of X onto a precompact subset of Y. notation $A \in K(X,Y)$

Remark: 1) Since A is linear this means that A maps every bounded set in X to a precompact subset of Y.

2) By the sequence criterion for compactness of metric space (i.e. $E \subset \mathcal{M}$ is compact iff $\forall (x_n)_{n\in\mathbb{N}} \subset E$ bounded has a convergent subsequence)

$\Longrightarrow$ A is compact iff for each bounded sequence $(x_n)_{n\in\mathbb{N}}$ the sequence $(Ax_n)_{n\in\mathbb{N}}$ has a convergent subsequence.

THM A compact operator is bounded i.e. $K(X,Y) \subset B(X,Y)$.

Proof. If $A: X \to Y$ is compact and B is the closed unit ball in X, then $\overline{A(B)}$ is compact therefore bounded. $\Rightarrow$ There is $M>0$ s.t. $\|y\| \leq M$ for all $y \in \overline{A(B)}$. $\Rightarrow$ $\|Ax\| \leq M$ for $\|x\| \leq 1$

$$\Downarrow$$

$$\|A\| \leq M$$

$\circ$!

Remark. So the continuity is an immediate consequence of the definition $\Rightarrow$ in some books compact operators ~~calle~~ are called <u>completely continuous</u> for Hilbert spaces (we will see).

Def. An operator T is called <u>finite rank operator</u> if $\operatorname{Ran} T$ is finite dimensional.

Prop. Suppose $K \in B(\mathcal{H}_1, \mathcal{H}_2)$ is of rank n. There exists vectors $v_1, v_2, \dots, v_n \in \mathcal{H}_1$ and $\varphi_1, \varphi_2, \dots, \varphi_n \in \mathcal{H}_2$ s.t. for every $x \in \mathcal{H}_1$

$$Kx = \sum_{i=1}^{n} \langle x, v_i \rangle \varphi_i .$$

The vectors $\varphi_1, \dots, \varphi_n$ may be chosen to be any ONB for $\operatorname{Ran} K$.

Proof: Let $\varphi_1, \dots, \varphi_n$ be an ONB for $\operatorname{Ran} K$.

$\hookrightarrow \forall x \in \mathcal{H}_1$

$$Kx = \sum_{i=1}^{n} \langle Kx, \varphi_i \rangle \varphi_i .$$

For each i, $f_i(x) := \langle Kx, \varphi_i \rangle$ defines a bounded linear functional on $\mathcal{H}_1$

$(f_i \in \mathcal{H}_1^*)$

Riesz representation thm $\Longrightarrow$

$\exists!\ v_i \in \mathcal{H}_1$ st $\forall x \in \mathcal{H}_1$

$$\langle Kx, \varphi_i \rangle = f_i(x) = \langle x, v_i \rangle \qquad 1 \le i \le n$$

$$\Rightarrow \quad Kx = \sum_{i=1}^{n} \langle x, v_i \rangle \varphi_i .$$

$\circ$!

Prop: ~~If $K \in B$~~ Every finite rank operator K is compact.

Proof: Let $(x_n)_{n \in \mathbb{N}}$ be a bounded sequence in $X \Rightarrow (Kx_n)_{n\in\mathbb{N}}$ is bounded in $\operatorname{Ran} K$. $\operatorname{Ran} K$ is isometric to $\mathbb{C}^d$ for some $d \Rightarrow (Kx_n)_{n\in\mathbb{N}}$ has a convergent subsequence.

$\circ$!

Corollary: $\circ$ If $\dim \mathcal{H}_1 < \infty$, then any linear map $K : \mathcal{H}_1 \to \mathcal{H}_2$ is bounded (we showed)

Moreover: $\circ$ Since $\operatorname{Ran} K = K\mathcal{H}_1$ is finite dimensional

$\Downarrow$ a)

K is compact

<u>Example:</u> The identity operator on an infinite dimensional Hilbert space is <u>not compact</u>.

! $\varphi_1, \varphi_2, \ldots$ ON set in $\mathcal{H}$

$$\Rightarrow \quad \| I\varphi_n - I\varphi_m \| = \| \varphi_n - \varphi_m \| = \sqrt{\langle \varphi_n - \varphi_m, \varphi_n - \varphi_m \rangle} =$$

$$= \{ \underbrace{\|\varphi_n\|^2}_{=1} + \underbrace{\|\varphi_m\|^2}_{=1} - \underbrace{\langle \varphi_n, \varphi_m \rangle}_{=0} - \underbrace{\langle \varphi_m, \varphi_n \rangle}_{=0} \}^{1/2} = \sqrt{2}$$

$\Rightarrow$ $(I\varphi_n)_{n\in\mathbb{N}}$ has no convergent subsequence even though $\{\varphi_n\}_{n\in\mathbb{N}}$ is bounded.

<u>Prop:</u> $K(X,Y)$ is a closed subspace of $B(X,Y)$.

<u>Proof.</u> It follows easily from the definition that a linear combination of compact operators is also compact.

Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of compact operators converging in norm to a bounded operator A: $\|A_n - A\| \to 0$.

Given $\varepsilon > 0$ there is $n \in \mathbb{N}$ s.t. $\|A_n - A\| < \frac{\varepsilon}{2}$

Let S be the unit ball of X.

A_n is compact $\Rightarrow$ $A_n(S) \subset Y$ can be covered by a finite number of balls of radius $\varepsilon/2$

Keeping the same centres and increasing the radii to ε we get a finite collection of balls that covers $A(S)$

$\Downarrow$

$A(S)$ is precompact.

$\square$

135)

~~THM~~ Corollary If $A \in B(X,Y)$ and there exists a sequence (A_n) of finite rank operators s.t. $\|A_n - A\| \to 0$ then $A \in K(X,Y)$.

Remark. i.e. $T \in B(X,Y)$ is compact if it is the limit of sequence of finite rank operators.

THM Let A and B be bounded operators. If either A or B is compact, then the product AB is compact.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence. Then $(Bx_n)_{n\in\mathbb{N}}$ is bounded and if A is compact, then $(ABx_n)_{n\in\mathbb{N}}$ has a convergent subsequence $\Rightarrow$ AB is compact.
If B is compact then $(Bx_n)_{n\in\mathbb{N}}$ has a convergent subsequence. The image of this subsequence under A is convergent, since A is continuous.

o!

THM: $A \in K(\mathcal{H}_1, \mathcal{H}_2)$ iff $A^* \in K(\mathcal{H}_2, \mathcal{H}_1)$.

Proof. Suppose A is compact, let $(x_n)_{n\in\mathbb{N}} \subset \mathcal{H}_2$ is bounded, namely $\|x_n\| = 1$
$A^* \in B(\mathcal{H}_2, \mathcal{H}_1)$ $\Rightarrow$ AA^* is compact $\Rightarrow$ there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ s.t.

$(AA^* x_{n_k})_{k\in\mathbb{N}}$ converges.

$\Downarrow$

$$\|A^* x_{n_k} - A^* x_{n_l}\|^2 = \langle AA^*(x_{n_k} - x_{n_l}), x_{n_k} - x_{n_l}\rangle \le$$

$$\underset{\text{C-S}}{\le} \|AA^*(x_{n_k} - x_{n_l})\| \cdot \|x_{n_k} - x_{n_l}\| \underset{\|x_n\|=1}{\le} 2\|AA^* x_{n_k} - AA^* x_{n_l}\| \xrightarrow{(*)} 0$$

as $k, l \to \infty$.

$\Downarrow$

$(A^* x_{n_k})_{k\in\mathbb{N}}$ is Cauchy sequence in $\mathcal{H}_1$, but $\mathcal{H}_1$ is complete

$\Downarrow$

convergent

$\Downarrow$

A^* is compact.

If A^* is compact $\Rightarrow$ $A^{**} = (A^*)^* = A$ is compact.

☺!

<u>Corollary</u>: $K(\mathcal{H})$ is a closed $*$-ideal in $\mathcal{B}(\mathcal{H})$.

ie:
- $K(\mathcal{H})$ is a closed subspace of $\mathcal{B}(\mathcal{H})$
- $K(\mathcal{H})$ is closed under the operation of taking adjoint
- $A \in K(\mathcal{H}),\ B \in \mathcal{B}(\mathcal{H}) \Rightarrow AB, BA \in K(\mathcal{H})$

(two sided ideal).

Examples.

(1) Let $(\lambda_n)_{n\in\mathbb{N}}$ be a complex sequence s.t. $\lambda_n \to 0$ $(n\to\infty)$.
Define $K \in \mathcal{B}(\ell_2)$ by

$$K(\alpha_1,\alpha_2,\ldots) = (\lambda_1\alpha_1, \lambda_2\alpha_2,\ldots) \qquad \alpha = (\alpha_1,\alpha_2,\ldots)\in\ell_2.$$

For each $n\in\mathbb{N}$ let $K_n \in \mathcal{B}(\ell_2)$ be

$$K_n(\alpha_1,\alpha_2,\ldots) := (\lambda_1\alpha_1, \lambda_2\alpha_2,\ldots,\lambda_n\alpha_n, 0, 0,\ldots)$$

$\Rightarrow$ K_n is of finite rank $\Rightarrow$ K_n is compact.

Since $\|K_n - K\| \le \sup_{k\ge n} |\lambda_k| \to 0$ as $n\to\infty$

$\Rightarrow$ K is compact.

(2) Given the infinite matrix $(a_{ij})_{i,j=1}^{\infty}$, where $\sum_{i,j=1}^{\infty} |a_{ij}|^2 < \infty$
let $A \in \mathcal{B}(\ell_2)$ be the operator corresponding to this matrix.
Let $A_n \in \mathcal{B}(\ell_2)$, $n\in\mathbb{N}$ be corresponding to the matrix

$$(a_{ij}^{(n)})_{i,j=1}^{\infty}, \text{ where } \quad a_{ij}^{(n)} = \begin{cases} a_{ij} & , 1\le i,j\le n \\ 0 & , \text{otherwise.} \end{cases}$$

$\Rightarrow$ A_n is of finite rank $\Rightarrow$ A_n is compact

truncated matrix

$\|A - A_n\| \xrightarrow[n\to\infty]{} 0$ $\Rightarrow$ A is compact.

(3) Let $T_K : L^2([a,b]) \to L^2([a,b])$ be the integral operator:

$$(T_K f)(t) = \int_a^b K(t,s) f(s)\, ds \quad , K \in L^2([a,b]\times[a,b])$$

$\Rightarrow$ T_K is bounded $\|T_K\| \leq \left(\int_a^b \int_a^b |K(t,s)|^2 ds\, dt\right)^{1/2} = \|K\|$.

We construct a sequence of operators of finite rank which converges to K in norm:

Let $\varphi_1, \varphi_2, \dots$ be an ONB for $L^2([a,b])$.

$$\Phi_{ij}(t,s) := \varphi_i(t)\overline{\varphi_j(s)} \qquad i,j = 1,2,\dots$$

is an ONB for $L^2([a,b]\times[a,b])$ (check it!)

$$\Rightarrow \quad K = \sum_{i,j=1}^{\infty} \langle K, \Phi_{ij}\rangle \Phi_{ij} .$$

Define

$$K_n(t,s) := \sum_{i,j=1}^{n} \langle K, \Phi_{ij}\rangle \Phi_{ij}(t,s) .$$

$\Rightarrow$ $\|K - K_n\| \longrightarrow 0$ as $n \to \infty$.

Let T_{K_n} be the integral operator

$$(T_{K_n} f)(t) = \int_a^b K_n(t,s) f(s)\, ds .$$

$\Rightarrow$ T_{K_n} is a finite rank operator since $\operatorname{Ran} T_{K_n} \subset \operatorname{Span}\{\varphi_1, \dots, \varphi_n\}$

and $\|T_K - T_{K_n}\| \leq \|K - K_n\| \longrightarrow 0$ $\Rightarrow$ T_K is <u>compact</u>.

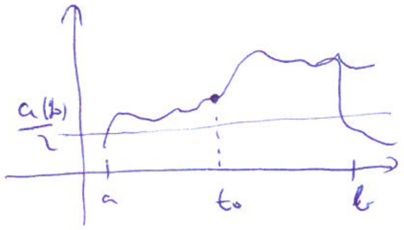
141)

(5) Given a complex valued function ~~a(t)~~ $a \in C[a,b]$, let $A: L^2([a,b]) \to L^2([a,b])$ be the bounded linear op:

$$(Af)(t) := a(t)f(t) .$$

Suppose that $a(t_0) \neq 0$ for some $t_0 \in [a,b]$.

$a(t)$ is continuous $\Longrightarrow$ $|a(t)| \geq \frac{|a(t_0)|}{2} > 0$ for all t in some compact interval J containing t_0



Let $\tilde{\varphi}_1, \tilde{\varphi}_2, \ldots$ be an ONB for $L^2(J)$.

$$\varphi_n := \chi_J \cdot \tilde{\varphi}_n \quad \Longrightarrow \quad \|\varphi_n\| = 1$$

for $n \neq m$

$$\|A\varphi_n - A\varphi_m\|^2 = \int_a^b |a(t)|^2 |\varphi_n(t) - \varphi_m(t)|^2 \, dt \geq \frac{|a(t_0)|^2}{4} \underbrace{\int_J |\tilde{\varphi}_n(t) - \tilde{\varphi}_m(t)| \, dt}_{=2}$$

$$= \frac{|a(t_0)|^2}{2}$$

$\Longrightarrow$ $(A\varphi_n)_{n \in \mathbb{N}}$ does not have a convergent subsequence

$\Downarrow$

A is <u>not compact</u>.

Remark.

$A : X \to Y$ linear is bounded iff it is continuous

$\Updownarrow$

A is bounded iff it maps every convergent sequence in X to a convergent sequence in Y.

ie: $$x_n \to x \implies Ax_n \to Ax$$

here $x_n \to x \iff \|x_n - x\| \to 0$ norm or strong convergence

Let suppose that $(x_n)_{n\in\mathbb{N}} \subset X$ converges weakly to x ie

$\forall \ \varphi \in X^* \quad \varphi(x_n - x) \to 0$ as $n \to \infty$

$\Rightarrow$ for simplicity, if $X = \mathcal{H}$ Hilbert space

$\iff \forall \ v \in \mathcal{H} \quad \langle v, x_n - x \rangle \to 0$ as $n \to \infty$

$\Rightarrow A \in \mathcal{B}(\mathcal{H}) \quad \langle v, A(x_n - x)\rangle = \langle A^* v, x_n - x\rangle \to 0$

Thus for every bounded operator A

$$x_n \xrightarrow{w} x \implies Ax_n \xrightarrow{w} Ax.$$

A stronger requirement:

Def: A is completely continuous if $x_n \xrightarrow{w} x \implies Ax_n \to Ax$

THM: Every compact operator A is completely continuous.

Proof: Let $x_n \xrightarrow{w} x$. Then $(\|x_n\|)_{n\in\mathbb{N}}$ is bounded.

If (Ax_n) does not converge strongly to Ax, then there exists an $\varepsilon > 0$ and a subsequence $(x_{m'})$ s.t.

$\|Ax_{m'} - Ax\| \geq 0$ for all m'.

Since $(x_{m'})$ is bounded and A is compact, $(Ax_{m'})$ has a convergent subsequence.

Suppose y is the limit of this sequence, then y is also its weak limit, but for any bounded operator

$$x_n \xrightarrow{w} x \text{ implies } Ax_n \xrightarrow{w} Ax, \text{ so}$$

$$y = Ax \quad : \quad \lightning$$

$\circ$!

In Hilbert spaces the converse is also true:

THM: If A is a completely continuous operator on a Hilbert space, then A is compact.

Remark: More generally, if X is a reflexive Banach space, then every completely continuous operator on it is compact.