Stochastics
Problem sheet 4 - Branching processes
Fall 2021

2. The teacher gives Jack problems to solve. Jack can solve each problem correctly with probability 1/2, independently from the others. For each incorrect solution, the teacher gives him two more problems to solve. Jack starts with a single problem to solve. What is the probability that Jack will eventually finish solving the problems? What is the expected number of problems he has to solve before he finishes?

   Solution. Define a branching process where individuals are the problems Jack is given, and each problem has 2 offspring if Jack's solution is wrong and 0 if the solution is correct.

   $G(z) = \frac{1}{2} + \frac{1}{2}z^2$, and $G'(1) = 1$, so this is a critical process. That means that the probability of extinction is 1, so Jack will eventually finish with probability 1, but $\mathbb{E}(N) = \infty$, so the expected number of problems he has to solve is infinite.

3. Let $\Theta(p)$ denote the probability of extinction for a branching process with offspring distribution PGEO($p$). Calculate the function $\Theta(p)$ for $0 < p < 1$.

   Solution.
$$G(z) = \sum_{k=0}^{\infty} p(1-p)^k z^k = \frac{p}{1-(1-p)z}.$$
   The solutions of $G(z) = z$ are 1 and $\frac{p}{1-p}$. For $p < 1/2$, the smallest nonnegative root is $\frac{p}{1-p}$, otherwise it is 1, so
$$\Theta(p) = \begin{cases} \frac{p}{1-p} & \text{for } 0 < p < 1/2 \\ 1 & \text{for } 1/2 \leq p < 1 \end{cases}$$

6. A nuclear reactor contains a large number of nuclei that split when hit by a neutron. The fate of a neutron in the reactor may be the following:

   - it is absorbed in a non-splitting nucleus with probability $p$, or
   - it splits a nucleus; the original neutron is absorbed, but 1, 2 or 3 new neutrons are released with equal probability $\frac{1-p}{3}$.

   The fate of each neutron is independent from all other neutrons and also from the past. The value of $p$ depends on the size and shape of the reactor and also on the use of regulatory devices.

   We fire a single neutron in the reactor. Let generation 1 be the neutrons released by the original single neutron (if there are any), generation 2 are the neutrons released by the neutrons in generation 1 and so on. The number of neutrons in generation $k$ is denoted by $X_k$. $N$ denotes the total number of neutrons in the reactor throughout the process.

   Answer the following questions for both $p = 1/4$ and $p = 5/8$.

   (a) Calculate the generating function of $X_1$ and $X_2$.
   (b) $\mathbb{E}X_1 =?$ $\mathbb{E}X_{10} =?$
   (c) $\mathbb{P}(X_4 = 0) =?$

(d) $\mathbb{P}(N < \infty) =?$ $\mathbb{E}N =?$

Solution.

(a) The generating function of $X_1$ is

$$G(z) = p + \frac{1-p}{3}z + \frac{1-p}{3}z^2 + \frac{1-p}{3}z^3,$$

and the generating function of $X_2$ is $G(G(z))$.

(b)

$$\mathbb{E}X_1 = G'(1) = 2(1-p) = \begin{cases} 1.5 & \text{for } p = 1/4, \\ 0.75 & \text{for } p = 5/8, \end{cases}$$

and

$$\mathbb{E}X_{10} = (G'(1))^{10} \approx \begin{cases} 57.665 & \text{for } p = 1/4, \\ 0.0563 & \text{for } p = 5/8. \end{cases}$$

(c)

$$\mathbb{P}(X_4 = 0) = G(G(G(G(0)))) \approx \begin{cases} 0.389 & \text{for } p = 1/4, \\ 0.904 & \text{for } p = 5/8. \end{cases}$$

(d) For $p = 5/8$, the process is subcritical, so $\mathbb{P}(N < \infty) = 1$, and $\mathbb{E}N = \frac{1}{1-G'(1)} = 4$. For $p = 1/4$, the process is supercritical, so $\mathbb{E}N = \infty$, and $\mathbb{P}(N < \infty) = z^*$, where $z^*$ is the smallest nonnegative solution of $G(z) = z$. Since $z = 1$ is always a solution, $G(z) - z$ can be factored as

$$G(z) - z = \frac{1}{3}(z-1)(-3p + (2-p)z + (1-p)z^2) = 0.$$

For $p = 1/4$, the solutions are

$$z_1 = 1, \qquad z_2 = \sqrt{2} - 1 \approx 0.414, \qquad z_3 = -1 - \sqrt{2} \approx -2.414,$$

so $\mathbb{P}(N < \infty) = z^* = z_2 \approx 0.414$.

7. In medieval times, a nobleman has a number of sons with offspring distribution PGEO(0.4) who carry on his family name. Then each of his male children will have a random number of sons (who carry on the family name) with the same distribution, independent from everybody else, and so on.

(a) Model the scenario with a branching process. What is the expectation of the offspring distribution? Is the process subcritical, critical or supercritical?

(b) What is the expected number of the male grandchildren of the original nobleman?

(c) What is the probability that the nobleman's family name dies out eventually?

Solution.

(a) Individuals in the branching process are the nobleman's male descendants, and the offspring of an individual are simply his male children. For the offspring distribution PGEO(0.4), the generating function is

$$G(z) = \frac{0.4}{1 - 0.6z},$$

and $\mathbb{E}(X_1) = G'(1) = 1.5 > 1$, so the process is supercritical.

(b)

$$\mathbb{E}(X_2) = (G'(1))^2 = 2.25.$$

(c) The process is supercritical, so we need to solve $G(z) = z$. This leads to a quadratic equation whose solutions are $z_1 = 1$ and $z_2 = 2/3$, so ¶(extinction) $= z^* = 2/3$.

8. * Let $\mu > 1$, and consider the distribution of a branching process with offspring distribution POI($\mu$), *conditioned on extinction.* Prove that this has the same distribution as a branching process with offspring distribution POI($\lambda$), where $\lambda$ is the solution of $\lambda e^{-\lambda} = \mu e^{-\mu}$ for which $\lambda < 1$. This is known as the duality of the Poisson branching process. (Mind that the outcomes of the branching process are finite or infinite trees, and its distribution means the probability measure with which the branching process gives given trees as realizations.)

Solution. Both distributions can only take finite trees. Let $k \geq 1$ and $y_1, y_2, \ldots, y_k \geq 0$ be arbitrary integers. We want to prove

$$\begin{aligned}&\Pr_\mu(Y_1 = y_1, Y_2 = y_2, \ldots, Y_k = y_k | N < \infty) = \\ &\Pr_\lambda(Y_1 = y_1, Y_2 = y_2, \ldots, Y_k = y_k),\end{aligned}$$

where $\Pr_\mu(.|N < \infty)$ denotes the conditional distribution of a branching process with offspring distribution POI($\mu$) conditioned on extinction, while $\Pr_\lambda(.)$ denotes the distribution of a branching process with offspring distribution POI($\lambda$), and $Y_1, Y_2, \ldots$ are the number of offspring for each member of the population.

We remark that

$$y_1 + y_2 + \cdots + y_k = k - 1.$$

since everyone except the initial member has exactly one parent.

Our second remark is that

$$\Pr_\mu(N < \infty) = e^{\lambda - \mu};$$

this can be seen by checking $G(z) = z$ for $z = e^{\lambda-\mu}$:

$$G(e^{\lambda-\mu}) = e^{\mu(e^{\lambda-\mu}-1)} = e^{\mu(\lambda/\mu-1)} = e^{\lambda-\mu}$$

due to $\lambda/\mu = e^{\lambda-\mu}$ which was assumed in the problem. We also note that $\lambda < 1 < \mu$ implies $e^{\lambda-\mu} < 1$, so $e^{\lambda-\mu}$ is the smallest nonnegative solution of $G(z) = z$.

Based on these two remarks, we have

$$\begin{aligned}&\Pr_\lambda(Y_1 = y_1, Y_2 = y_2, \ldots, Y_k = y_k) = \frac{\lambda^{y_1}}{y_1!}e^{-\lambda}\frac{\lambda^{y_2}}{y_2!}e^{-\lambda}\cdots\frac{\lambda^{y_k}}{y_k!}e^{-\lambda} = \\ &\frac{\lambda^{y_1+\cdots+y_k}}{y_1!y_2!\ldots y_k!}e^{-k\lambda} = \frac{\lambda^{k-1}}{y_1!y_2!\ldots y_k!}e^{-k\lambda}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\Pr_\mu(Y_1 = y_1, Y_2 = y_2, \ldots, Y_k = y_k | N < \infty) = \\ &\frac{\frac{\mu^{y_1}}{y_1!}e^{-\mu}\frac{\mu^{y_2}}{y_2!}e^{-\mu}\ldots\frac{\mu^{y_k}}{y_k!}e^{-\mu}}{\Pr_\mu(N < \infty)} = \frac{\mu^{y_1+\cdots+y_k}}{y_1!y_2!\ldots y_k!}e^{-k\mu}\cdot\frac{1}{\Pr_\mu(N < \infty)} = \\ &\frac{\mu^{k-1}}{y_1!y_2!\ldots y_k!}e^{-k\mu}\cdot\frac{1}{e^{\lambda-\mu}}.\end{aligned}$$

Then

$$\frac{\Pr_\lambda(Y_1 = y_1, Y_2 = y_2, \ldots, Y_k = y_k)}{\Pr_\mu(Y_1 = y_1, Y_2 = y_2, \ldots, Y_k = y_k | N < \infty)} =$$
$$\left(\frac{\lambda}{\mu}\right)^{k-1} e^{-k(\lambda-\mu)} \cdot e^{\lambda-\mu} = (*);$$

and using $\lambda/\mu = e^{\lambda-\mu}$ once again,

$$(*) = e^{(\lambda-\mu)(k-1)} e^{-k(\lambda-\mu)} \cdot e^{\lambda-\mu} = 1.$$