BÁLINT TÓTH:

# LIMIT THEOREMS OF PROBABILITY THEORY

## LECTURE NOTES

# CONTENTS

# I.
# STATIONARY SEQUENCES, ERGODIC THEOREMS

## STATIONARY SEQUENCES OF RANDOM VARIABLES

$(\Omega, \mathcal{F}, \mathbf{P})$ a probability space
$(S, \mathcal{S})$ a measurable space
$\xi_j : \Omega \to S$ measurable functions, $j \in \mathbb{N}$ (or $j \in \mathbb{Z}$)

**Definition.** *The sequence of ($S$-valued) random variables $\xi_j$ is* *stationary* *iff* $(\forall k \in \mathbb{N})$ *(or $(\forall k \in \mathbb{Z})$) and* $(\forall l \geq 0)$*:*

$$\textit{distrib}\,(\xi_0, \xi_1, \ldots, \xi_l) = \textit{distrib}\left(\xi_k, \xi_{k+1}, \ldots, \xi_{k+l}\right)$$

**Elementary remarks:**

1. A stationary sequence $\left(\xi_j\right)_{j\in\mathbb{N}}$ can always be embedded into a stationary sequence $\left(\xi_j\right)_{j\in\mathbb{Z}}$.

2. If $\left(\xi_j\right)_{j\in\mathbb{Z}}$ is a stationary sequence of $(S,\mathcal{S})$-valued random variables, $(\widetilde{S},\widetilde{\mathcal{S}})$ is another measurable space, $g : S^{\mathbb{Z}} \to \widetilde{S}$ is measurable map, and

$$\widetilde{\xi}_j := g(\dots,\xi_{j-1},\xi_j,\xi_{j+1},\dots)$$

Then: $\left(\widetilde{\xi}_j\right)_{j\in\mathbb{Z}}$ is a stationary sequence of $(\widetilde{S},\widetilde{\mathcal{S}})$-valued random variables

The essential content of ergodic theorems: generalizations of the laws of large numbers

If $\left(X_j\right)_{j=0}^{\infty}$ is a stationary sequence of $\mathbb{R}$-valued random variables, such that $\mathbf{E}\left( \left|X_j\right| \right) < \infty$, then

$$\frac{1}{n}\sum_{j=0}^{n-1} X_j \to \mathbf{E}\left( X_1 \right)$$

asymptotic time averages = state-space averages

– almost surely and in $L^1(\Omega, \mathcal{F}, \mathbf{P})$ (Birkhoff, difficult)

– in $L^2(\Omega, \mathcal{F}, \mathbf{P})$, (von Neumann, easier)

**Examples of stationary sequences:**

**Ex 1: I.i.d. sequences:**
$\left(\xi_j\right)_{j\in\mathbb{Z}}$ i.i.d. sequence of $(S,\mathcal{S})$-valued random variables.

**Ex 2: Finitely dependent sequences:**
Let $\left(\xi_j\right)_{j\in\mathbb{Z}}$ i.i.d. sequence of $(S,\mathcal{S})$-valued random variables, $(\widetilde{S},\widetilde{\mathcal{S}})$ another measurable space, $m \geq 0$ (fixed), $g : S^{m+1} \to \widetilde{S}$ measurable map. Then

$$\widetilde{\xi}_j := g(\xi_j, \ldots, \xi_{j+m})$$

is a $(\widetilde{S},\widetilde{\mathcal{S}})$-valued stationary sequence.
E.g. $\xi_j$ i.i.d. Bernoulli, $\widetilde{\xi}_j := \max\{\xi_j, \xi_{j+1}\}$.

**Ex 3a,3b:**

$\left(\xi_j\right)_{j\in\mathbb{Z}}$ i.i.d. Bernoulli, $\mathbf{P}\left(\xi_j = 0\right) = 1/2 = \mathbf{P}\left(\xi_j = 1\right)$.

$$\zeta_j := \sum_{k=0}^{\infty} 2^{-k-1}\xi_{j+k}$$

$$\eta_j := \sum_{k=0}^{\infty} 2^{-k-1}\xi_{j-k}$$

Then:

$$\text{distrib}\left(\zeta_j\right) = UNI[0,1] = \text{distrib}\left(\eta_j\right).$$

Remarks:

$$\zeta_{j+1} = \{2\zeta_j\} := 2\zeta_j - [2\zeta_j] \quad \text{deterministically!}$$

$\left(\eta_j\right)_{j\geq 0}$ is a Markov chain on $[0,1]$.

**Ex 4: Stationary Markov chains**

Let $S$ be a finite or countable state space, $P = \left(P_{\alpha,\beta}\right)_{\alpha,\beta\in S}$ stochastic matrix, $\pi : S \to [0,1]$, $\sum_{\alpha\in S}\pi(\alpha) = 1$ stationary for $P$:

$$\sum_{\alpha\in S}\pi(\alpha)P_{\alpha,\beta} = \pi(\beta).$$

$\left(\xi_j\right)_{j\geq 0}$ the stationary Markov chain:

$$\mathbf{P}\Big(\xi_0 = \alpha_0, \xi_1 = \alpha_1, \ldots, \xi_l = \alpha_l\Big) = \pi(\alpha_0)P_{\alpha_0,\alpha_1}\ldots P_{\alpha_{l-1},\alpha_l}$$

**Ex 5:**
**Rotations of the circle:** $S = [0,1)$, $\mathcal{S} =$ Borel, $\mathbf{P} =$ Lebesgue.

$$\theta \in (0,1) \text{ (fixed)}, \qquad \xi_j(\omega) := \{\omega + \theta\}, \quad j \in \mathbb{Z}$$

**Ex 6:**
**"Bernoulli shift":** (see also Ex 3a) $S = [0,1)$, $\mathcal{S} =$ Borel, $\mathbf{P} =$ Lebesgue.

$$\xi_j(\omega) := \{2^j\omega\} = 2^j\omega - [2^j\omega], \qquad j \geq 0$$

## MEASURE PRESERVING TRANSFORMATIONS, DYNAMICAL SYSTEMS

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. The $T : \Omega \to \Omega$ measurable transformation is measure preserving if

$$\forall A \in \mathcal{F} : \qquad \mathbf{P}\Big(T^{-1}A\Big) = \mathbf{P}\Big(A\Big)$$

We call $(\Omega, \mathcal{F}, \mathbf{P}, T)$ an endomorphism or a dynamical system. If $T$ is a.s. invertible we call it an automorphism

Let $(S, \mathcal{S})$ be another measurable space and $g : \Omega \to S$ a measurable function. Then

$$\xi_j := g(T^j \omega)$$

is a stationary sequence of $S$-valued random variables.

**Remark:** Any stationary sequence of random variables can be realized this way!

$(S, \mathcal{S})$ measurable space, $\left(\xi_j\right)_{j=0}^{\infty}$ stationary sequence of $S$-valued random variables.

$$\Omega := S^{\mathbb{N}} = \{\omega = (\omega_0, \omega_1, \omega_2, \dots) : \omega_j \in S\}$$

$$\mathcal{F} := \sigma\Big(\mathcal{S} \times \mathcal{S} \times \mathcal{S} \times \dots\Big)$$

$$\mathbf{P} = \text{joint distribution of } \left(\xi_j\right)_{j=0}^{\infty}$$

$$T : \Omega \to \Omega, \qquad \Big(T\omega\Big)_j = \omega_{j+1}$$

$$g : \Omega \to S, \qquad g(\omega) := \omega_0$$

**The invariant sigma-algebra, ergodicity:**

Let $(\Omega, \mathcal{F}, \mathbf{P}, T)$ be an endomorphism. Then

$$\mathcal{I} := \{A \in \mathcal{F} : \mathbf{P}\Big(A \circ T^{-1}A\Big) = 0\} \subset \mathcal{F}$$

is the sub-sigma-algebra of invariant sets.

**Definition.** *The dynamical system* $(\Omega, \mathcal{F}, \mathbf{P}, T)$ *is ergodic iff the invariant sigma-algebra* $\mathcal{I}$ *is trivial with respect to* $\mathbf{P}$*:*

$$\forall A \in \mathcal{I} : \qquad \mathbf{P}\Big(A\Big) \in \{0, 1\}.$$

*Equivalently:* $(\Omega, \mathcal{F}, \mathbf{P}, T)$ is ergodic iff for $f : \Omega \to \mathbb{R}$ measurable

$$\Big\{f(T\omega) = f(\omega) \text{ a.s.}\Big\} \Leftrightarrow \Big\{f(\omega) = \text{const. a.s.}\Big\}$$

**Ex1: I.i.d. sequence:** $(S, \mathcal{S}, \mathbf{P}_1)$ a probability space,

$$\Omega := S^{\mathbb{N}} = \{\omega = (\omega_0, \omega_1, \omega_2, \dots) : \omega_j \in S\}$$

$$\mathcal{F} := \sigma\Big(\mathcal{S} \times \mathcal{S} \times \mathcal{S} \times \dots\Big)$$

$$\mathbf{P} = \mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1 \times \dots$$

$$T : \Omega \to \Omega, \qquad \Big(T\omega\Big)_j = \omega_{j+1}$$

**Theorem.** *The endomorphism* $(\Omega, \mathcal{F}, \mathbf{P}, T)$ *is ergodic.*

*Proof.* The tail sigma-algebra is

$$\mathcal{T} := \bigcap_n \sigma\left(\omega_n, \omega_{n+1}, \omega_{n+2}, \dots\right)$$

Fact: $\mathcal{I} \subset \mathcal{T}$. Not very difficult.
Kolmogorov's 0-1 law: $\mathcal{T}$ is $\mathbf{P}$-trivial. □

**Ex 2, 3: Factors:**
$(\Omega, \mathcal{F}, \mathbf{P}, T)$ and $(\widetilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}}, \tilde{T})$ dynamical systems, $\varphi : \Omega \to \widetilde{\Omega}$ measurable, such that

$$\mathbf{P}(\varphi^{-1}(A)) = \tilde{\mathbf{P}}(A) \qquad \forall A \in \tilde{\mathcal{F}}$$

$$\varphi \circ T = \tilde{T} \circ \varphi \qquad \mathbf{P}-\text{a.s.}$$

then $(\widetilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}}, \tilde{T})$ is a factor of $(\Omega, \mathcal{F}, \mathbf{P}, T)$.

**Theorem.** *If* $(\widetilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}}, \tilde{T})$ *is a factor of* $(\Omega, \mathcal{F}, \mathbf{P}, T)$ *and* $(\Omega, \mathcal{F}, \mathbf{P}, T)$ *is ergodic then so is* $(\widetilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}}, \tilde{T})$.

*Proof.* Homework □

**Ex 4: Ergodic Markov chains:**

The state space: $(S, \mathcal{S})$ finite or countable

The stochastic matrix $P = \Big(P_{\alpha,\beta}\Big)_{\alpha,\beta \in S}$,

$\pi$ probability measure on $S$, stationary for $P$: $\pi P = \pi$.

$$\Omega := S^{\mathbb{N}} = \{\omega = (\omega_0, \omega_1, \omega_2, \dots) : \omega_j \in S\}$$

$$\mathcal{F} := \sigma\Big(\mathcal{S} \times \mathcal{S} \times \mathcal{S} \times \dots\Big)$$

$$\mathbf{P}\Big(\omega_0, \omega_1, \dots, \omega_l\Big) = \pi(\omega_0) P_{\omega_0,\omega_1} \dots P_{\omega_{l-1},\omega_l}$$

$$T : \Omega \to \Omega, \qquad \Big(T\omega\Big)_j = \omega_{j+1}$$

**Theorem.** *The dynamical system $(\Omega, \mathcal{F}, \mathbf{P}, T)$ is ergodic iff $P$ is irreducible.*

*Proof.* : Proof of $\Rightarrow$: trivial

Proof of $\Leftarrow$: Denote $\mathcal{F}_n := \sigma(\omega_0, \ldots, \omega_n)$ and let $A \in \mathcal{I}$.

Then $\mathbf{E}\left( 1\!\!1_A \,\middle|\, \mathcal{F}_n \right)$ is a bdd martingale w.r.t. the filtration $\mathcal{F}_n$ and

$$\mathbf{E}\left( 1\!\!1_A \,\middle|\, \mathcal{F}_n \right)(\omega) \stackrel{(1)}{=} \mathbf{E}\left( 1\!\!1_A \circ T^n \,\middle|\, \mathcal{F}_n \right)(\omega) \stackrel{(2)}{=} h(\omega_n)$$

(1): due to invariance of $A$

(2): due to the Markov property

Due to the martingale convergence theorem

$$h(\omega_n) = \mathbf{E}\left( 1\!\!1_A \,\middle|\, \mathcal{F}_n \right)(\omega) \xrightarrow{\text{a.s.}} \mathbf{E}\left( 1\!\!1_A \,\middle|\, \mathcal{F}_\infty \right)(\omega) = 1\!\!1_A(\omega)$$

This can hold only if $h \equiv$ const.

$\square$

**Ex 5: Rotations of the circle:**
$\Omega = [0,1)$, $\mathcal{F}$ = Borel, $\mathbf{P}$ = Lebesgue, $T\omega := \{\omega + \theta\}$

**Theorem.** *The dynamical system* $(\Omega, \mathcal{F}, \mathbf{P}, T)$ *is ergodic iff* $\theta$ *is irrational.*

*Proof.* Fourier method: let $f \in L^2(\Omega, \mathcal{F}, \mathbf{P})$.

$$f(\omega) \stackrel{L^2}{=} \sum_{k=-\infty}^{\infty} c_k e^{i2\pi k\omega}, \qquad c_k = \int_0^1 e^{-i2\pi k\omega} f(\omega) d\omega$$

Then

$$\Big\{ f(\omega) = f(T\omega) \quad \text{a.s.} \Big\} \Leftrightarrow \Big\{ \forall k \in \mathbb{Z} : \ c_k \left( e^{i2\pi k\theta} - 1 \right) = 0 \Big\}$$

$$\Leftrightarrow \left\{ \begin{array}{ll} \theta \notin \mathbb{Q} : & c_k = \delta_{k,0} \\ \theta = \frac{p}{q} \in \mathbb{Q} : & c_k = c_k \mathbb{1}_{\{k=mq\}} \end{array} \right\}$$

□

**Ex 6: "Bernoulli shift":**
$\Omega = [0,1)$, $\mathcal{F} =$ Borel, $\mathbf{P} =$ Lebesgue, $T\omega := \{2\omega\}$

**Theorem.** *The dynamical system* $(\Omega, \mathcal{F}, \mathbf{P}, T)$ *is ergodic.*

*Proof.* (See Ex1) Let $\widetilde{\Omega} = \{0,1\}^{\mathbb{N}}$, $\tilde{\mathcal{F}} = \dots$, $\tilde{\mathbf{P}} = \left(\frac{1}{2} : \frac{1}{2}\right)$-Bernoulli, $\tilde{T} =$ left shift

$$\varphi : \widetilde{\Omega} \to \Omega \qquad\qquad \varphi(\tilde{\omega}) := \sum_{j=0}^{\infty} 2^{-j-1}\tilde{\omega}_j$$

$$\varphi^{-1} : \Omega \to \widetilde{\Omega} \qquad\qquad \varphi^{-1}(\omega)_j := [2^j\omega] \bmod 2$$

Then $(\Omega, \mathcal{F}, \mathbf{P}, T) \overset{\varphi:1-1}{\longleftrightarrow} (\widetilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}}, \tilde{T})$, and $(\widetilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}}, \tilde{T})$ is ergodic, according to Ex1. □

Alternative proof: by Fourier method (Home work).

**Ex 7: Algebraic automorphism of the 2-d torus:**
$\Omega = [0,1) \times [0,1)$, $\mathcal{F} =$ Borel, $\mathbf{P} =$ Lebesgue,
$T(x, y) := (\{x + 2y\}, \{x + y\})$ (picture on blackboard)

**Ex 8: The "Baker's Transformation":**
$\Omega = [0,1) \times [0,1)$, $\mathcal{F} =$ Borel, $\mathbf{P} =$ Lebesgue,
$T(x, y) := (\{2x\}, \{2x + y/2\})$ (picture on blackboard)

In both examples:

**Theorem.** *The dynamical system* $(\Omega, \mathcal{F}, \mathbf{P}, T)$ *is ergodic.*

*Proof 1.* Fourier method
*Proof 2.* "Markov partition" □

**Ex 9: Statistical physics:**

$\Omega =$ phase space of physical particle system,

$\mathcal{F} =$ Borel,

$\mathbf{P} =$ Liouville measure

$=$ Lebesgue meas. restricted to manifold of conserved quantities,

$T_t :=$ Newtonian dynamical flow

**Theorem** (Liouville's theorem)**.** *The dynamical flow* $t \mapsto T_t$ *conserves the measure. I.e.* $(\Omega, \mathcal{F}, \mathbf{P}, T_t)$ *is a continuous time dynamical system.*

**Ludwig Boltzmann's ergodic hypothesis:** In physically relevant cases $(\Omega, \mathcal{F}, \mathbf{P}, T_t)$ is ergodic.

**Major open question! Answer known in very few cases.**

## KOOPMANISM AND VON NEUMANN'S (MEAN, $L^2$) ERGODIC THEOREM

$(\Omega, \mathcal{F}, \mathbf{P}, T)$: dynamical system,

$\mathcal{F} \supset \mathcal{I}$: its invariant sigma-algebra,

$\mathcal{H} := L^2(\Omega, \mathcal{F}, \mathbf{P})$: Hilbert space of square integrable functions,

$\mathcal{K} := L^2(\Omega, \mathcal{I}, \mathbf{P}) = \{f \in \mathcal{H} : f(T\omega) = f(\omega) \ \mathbf{P}\text{-a.s.}\}$: subspace of $T$-invariant $L^2$-functions.

Two linear operators:

$$\Pi : \mathcal{H} \to \mathcal{K}, \qquad \Pi f(\omega) := \mathbf{E}\Big( f \,\Big|\, \mathcal{I} \Big)(\omega)$$

$$U : \mathcal{H} \to \mathcal{H}, \qquad U f(\omega) := f(T\omega)$$

$\Pi$ is the orthogonal projection to the subspace $\mathcal{K}$

$U$ is Koopman's representation of the action $T$.

$\mathcal{K} = \mathrm{Ker}(U - I) = \{f \in \mathcal{H} : Uf = f\}$

**Lemma.** $U$ *is a (partial) isometry.*

*Proof.*

$$\begin{aligned}(Uf, Ug) &= \int_\Omega \overline{f(T\omega)} g(T\omega) d\mathbf{P}(\omega) \\ &\stackrel{(1)}{=} \int_\Omega \overline{f(\omega)} g(\omega) d\mathbf{P}(\omega) = (f, g)\end{aligned}$$

(1): due to invariance of the measure under the action $T$.

□

**Remark:** If $T$ is a.s. invertible then $U$ is unitary.

**Theorem** (von Neumann's mean ergodic theorem)**.** *Let*

$\mathcal{H}$*: a separable Hilbert space,*

$U \in \mathcal{B}(\mathcal{H})$*: a (partial) isometry,*

$\mathcal{K} := \mathrm{Ker}(U - I)$,

$\Pi$*: the orthogonal projection to the closed subspace* $\mathcal{K}$*.*

*Then*

$$\text{st-}\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} U^j = \Pi,$$

*That is,*

$$\forall f \in \mathcal{H} : \quad \lim_{n\to\infty} \left\| \frac{1}{n} \sum_{j=0}^{n-1} U^j f - \Pi f \right\| = 0$$

**Corollary.** $(\Omega, \mathcal{F}, \mathbf{P}, T)$: *a dynamical system,* $\mathcal{I}$: *its invariant sigma-algebra.*
*If* $f \in L^2(\Omega, \mathcal{F}, \mathbf{P})$ *then*

$$\lim_{n \to \infty} \int_\Omega \left| \frac{1}{n} \sum_{j=0}^{n-1} f(T^j \omega) - \mathbf{E}\Big( f \,\Big|\, \mathcal{I} \Big)(\omega) \right|^2 d\mathbf{P}(\omega) = 0.$$

*In particular, if* $(\Omega, \mathcal{F}, \mathbf{P}, T)$ *is ergodic then*

$$L^2\text{-}\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j \omega) = \int_\Omega f d\mathbf{P}.$$

*Proof.* Proof of von Neumann's mean ergodic theorem:

$$\mathcal{H} \overset{(1)}{=} \overline{\mathrm{Ran}(U-I)} \oplus \mathrm{Ker}(U^*-I)$$
$$\overset{(2)}{=} \overline{\mathrm{Ran}(U-I)} \oplus \mathrm{Ker}(U-I)$$

(1): $\forall A \in \mathcal{B}(\mathcal{H}):\quad \mathcal{H} = \overline{\mathrm{Ran}A} \oplus \mathrm{Ker}A^*$
(2): Since $U \in \mathcal{B}(\mathcal{H})$ is an isometry, $\mathrm{Ker}(U^*-I) = \mathrm{Ker}(U-I)$.
(Homework)

For $f \in \mathrm{Ker}(U-I)$:

$$Uf = f = \Pi f \qquad \Rightarrow \qquad \frac{1}{n}\sum_{j=0}^{n-1} U^j f = \Pi f$$

For $f \in \overline{\mathrm{Ran}(U - I)}$: $(\forall \varepsilon > 0)$ $(\exists g, h \in \mathcal{H})$ such that

$$\|h\| < \varepsilon \quad \text{and} \quad f = Ug - g + h.$$

Thus:

$$\frac{1}{n}\sum_{j=0}^{n-1} U^j f = \frac{1}{n}(U^n g - g) + \frac{1}{n}\sum_{j=0}^{n-1} U^j h$$

and hence

$$\left\| \frac{1}{n}\sum_{j=0}^{n-1} U^j f \right\| \leq \left( \frac{2}{n} + \varepsilon \right) \|g\|$$

□

## BIRKHOFF'S "INDIVIDUAL" (POINTWISE, ALMOST SURE) ERGODIC THEOREM

**Theorem** (Birkhoff's individual ergodic theorem)**.**

$(\Omega, \mathcal{F}, \mathbf{P}, T)$*: a dynamical system,* $\mathcal{I}$*: its invariant sigma-algebra.*

*If* $f \in L^1(\Omega, \mathcal{F}, \mathbf{P})$ *then*

$$\frac{1}{n} \sum_{j=0}^{n-1} f(T^j \cdot) \to \mathbf{E}\Big( f \Big| \mathcal{I} \Big)(\cdot)$$

$\mathbf{P}$*-a.s. and in* $L^1(\Omega, \mathcal{F}, \mathbf{P})$*.*

*In particular, if* $(\Omega, \mathcal{F}, \mathbf{P}, T)$ *is ergodic then*

$$\frac{1}{n} \sum_{j=0}^{n-1} f(T^j \cdot) \to \int_\Omega f d\mathbf{P}$$

$\mathbf{P}$*-a.s. and in* $L^1(\Omega, \mathcal{F}, \mathbf{P})$*.*

*Proof.* [Birkhoff 1931, Yosida & Kakutani 1939, Garsia 1965]

$$X_j = X_j(\omega) := f(T^j\omega), \qquad X := X_0,$$

$$S_k = S_k(\omega) := \sum_{j=0}^{k-1} X_j(\omega), \qquad S_0 = 0,$$

$$M_k = M_k(\omega) := \max\{S_j(\omega) : j = 0, 1, \ldots, k\}, \qquad M_0 = 0.$$

**Lemma** (The maximal ergodic lemma)**.**

$$\mathbf{E}\Big( X \mathbb{1}_{\{M_k > 0\}} \Big) \geq 0$$

*Explicitly spelled out:*

$$\int_\Omega f(\omega) \mathbb{1}_{\{M_k(\omega)>0\}} d\mathbf{P}(\omega) \geq 0$$

Mind the strict inequality: $M_k > 0$!

*Proof of the maximal lemma* (Garsia 1965)

$$\begin{aligned} X(\omega) &\overset{(1)}{=} \max\{S_j(\omega) : j = 1, \ldots, k+1\} - \max\{S_j(T\omega) : j = 0, \ldots, k\} \\ &\geq \max\{S_j(\omega) : j = 1, \ldots, k\} - \max\{S_j(T\omega) : j = 0, \ldots, k\} \\ &= \max\{S_j(\omega) : j = 1, \ldots, k\} - M_k(T\omega) \end{aligned}$$

(1): Since $S_{j+1}(\omega) = X(\omega) + S_j(T\omega), \quad j = 0, 1, \ldots.$

Hence

$$\int_\Omega X(\omega) 1\!\!1_{\{M_k(\omega)>0\}} d\mathbf{P}(\omega)$$

$$\geq \int_\Omega \Big(\max\{S_j(\omega) : j = 1,\ldots,k\} - M_k(T\omega)\Big) 1\!\!1_{\{M_k(\omega)>0\}} d\mathbf{P}(\omega)$$

$$\stackrel{(2)}{=} \int_\Omega \Big(M_k(\omega) - M_k(T\omega)\Big) 1\!\!1_{\{M_k(\omega)>0\}} d\mathbf{P}(\omega)$$

$$\geq \int_\Omega \Big(M_k(\omega) - M_k(T\omega)\Big) d\mathbf{P}(\omega) \stackrel{(3)}{=} 0.$$

(2): Here we use the strict inequality $M_k > 0$.
(3): Due to invariance of the measure under the action $T$.

□

*Proof of Birkhoff's theorem:*
Without loss of generality assume $\mathbf{E}\Big( f \,\Big|\, \mathcal{I} \Big) = 0$. Fix $\varepsilon > 0$ and define

$$L(\omega) := \limsup_{n\to\infty} \frac{S_n(\omega)}{n}, \qquad D^\varepsilon := \{\omega : L(\omega) > \varepsilon\} \in \mathcal{I},$$

$$X^\varepsilon(\omega) := \Big( X(\omega) - \varepsilon \Big) 1\!\!1_{D^\varepsilon}(\omega), \qquad S_k^\varepsilon(\omega) := \sum_{j=0}^{k-1} X_j^\varepsilon(\omega),$$

$$M_k^\varepsilon(\omega) := \max\{S_j^\varepsilon(\omega) : j = 0, \dots, k\}, \qquad F^\varepsilon := \cup_k \{\omega : M_k^\varepsilon(\omega) > 0\}.$$

Note that

$$F^\varepsilon = \{\omega : \sup_k M_k^\varepsilon(\omega) > 0\} = \{\omega : \sup_k S_k^\varepsilon(\omega) > 0\} = D^\varepsilon$$

$$0 \overset{(1)}{\leq} \mathbf{E}\Big( X^{\varepsilon} 1\!\!1_{\{M_n^{\varepsilon} > 0\}} \Big) \overset{(2)}{\rightarrow} \mathbf{E}\Big( X^{\varepsilon} 1\!\!1_{F^{\varepsilon}} \Big)$$

$$\overset{(3)}{=} \mathbf{E}\Big( X^{\varepsilon} 1\!\!1_{D^{\varepsilon}} \Big) \overset{(4)}{=} \mathbf{E}\Big( (X - \varepsilon) 1\!\!1_{D^{\varepsilon}} \Big) \overset{(5)}{=} -\varepsilon \mathbf{P}\Big( D^{\varepsilon} \Big)$$

(1): due to the maximal lemma
(2): dominated convergence
(3): since $F^{\varepsilon} = D^{\varepsilon}$
(4): by definition of $X^{\varepsilon}$
(5): since $D^{\varepsilon} \in \mathcal{I}$ and $\mathbf{E}\Big( X \Big| \mathcal{I} \Big) = 0$.

It follows that $\forall \varepsilon > 0 :\ \mathbf{P}\Big( D^{\varepsilon} \Big) = 0$, and

$$\mathbf{P}\Big( L > 0 \Big) = \mathbf{P}\Big( \cup_{\varepsilon > 0} D^{\varepsilon} \Big) = \lim_{\varepsilon \to 0} \mathbf{P}\Big( D^{\varepsilon} \Big) = 0.$$

□

## BACK TO THE EXAMPLES

**Ex 1: I.i.d. sequence**: $X_j$, i.i.d., $\mathbf{E}\left( \left| X_j \right| \right) < \infty$

$$\frac{1}{n} \sum_{j=0}^{n-1} X_j \to \mathbf{E}\left( X_j \right)$$

Laws of large numbers.

**Ex 2, 3: Factors:** Laws of large numbers for factors of i.i.d. sequences.

**Ex 4: Stationary denumerable Markov chains:**

$\xi_j$: stationary MC on $S = \cup_m S^{(m)}$, ($S^{(m)}$ irred. comp.)

$$f : S \to \mathbb{R} : \qquad \sum_{\alpha \in S} \pi(\alpha)\,|\,f(\alpha)\,| < \infty$$

$$\frac{1}{n}\sum_{j=0}^{n-1} f(\xi_j) \to \sum_m 1\!\!1_{\{\xi_0 \in S^{(m)}\}} \frac{\sum_{\alpha \in S^{(m)}} \pi(\alpha) f(\alpha)}{\sum_{\alpha \in S^{(m)}} \pi(\alpha)}$$

Law of large numbers for MC

**Ex 5: Rotations of the circle:** $\theta \notin \mathbb{Q}$, $f \in L^1([0,1), \mathcal{B}, d\omega)$:

$$\frac{1}{n} \sum_{j=0}^{n-1} f(\cdot + j\theta) \to \int_0^1 f(\omega) d\omega, \qquad \text{a.s. and in } L^1.$$

**Remark:** For $f := 1\!\!1_{[a,b)}$ stronger:

$$\forall \omega \in [0,1): \qquad \frac{1}{n} \sum_{j=0}^{n-1} 1\!\!1_{[a,b)}(\omega + j\theta) \to b - a.$$

Proof: Homework.

**A consequence:**

Fix $k \in \{1, 2, \ldots, 9\}$. Then

$$\frac{\#\{m < n : 2^m = k \cdots \text{ in dec.}\}}{n} \to \frac{\log(k+1) - \log k}{\log 10}$$

*Proof.* Let $\theta := \frac{\log 2}{\log 10} \notin \mathbb{Q}$.

$\left\{2^m = k \cdots \text{ in dec.}\right\} \Leftrightarrow \left\{\{m\theta\} \in A_k := [\log k / \log 10, \log(k+1)/\log 10)\right\}$

□

**Ex 6: Bernoulli shift:**

$\omega \in [0,1)$, binary expansion: $\omega = \sum_{j=1}^{\infty} \omega_j 2^{-j}$

**Theorem.** *For Lebesgue-a.e.* $\omega \in [0,1)$ *any fixed* $\{0,1\}$ *string* $(\epsilon_1, \epsilon_2, \ldots, \epsilon_k)$ *occurs with its natural proper density* $2^{-k}$.

I.e. "Almost all real numbers are normal."

**Statistical physics:**

$$\text{Ergodicity}$$
$$\Updownarrow$$
$$\left\{ \text{time averages} \;=\; \text{phase space averages} \right\}$$

At the heart of statistical physics.

# II.
# CONVERGENCE IN DISTRIBUTION, WEAK CONVERGENCE

## CONVERGENCE IN DISTRIBUTION, BASICS

$(S, d)$ complete, separable metric space,
$\mathcal{S}$ its Borel-sigma-algebra
e.g. $\mathbb{R}$, $\mathbb{R}^n$ with Euclidean distance,
$C([0,1])$, $C([0,\infty))$ with sup-norm distance

**Definition.** *A probability measure* $\nu$ *on* $(S, \mathcal{S})$ *is regular if* $(\forall A \in \mathcal{S})$

$$\nu\Big(A\Big) = \sup\{\nu\Big(K\Big) : K \subseteq A,\ K \text{ compact}\} = \inf\{\nu\Big(O\Big) : A \subseteq O,\ O \text{ open}\}$$

All measures considered will be assumed regular.

$\mu_n$, $n = 1, 2, \ldots$ and $\mu$ regular probability measures on $(S, \mathcal{S})$.

$Y_n$, $n = 1, 2, \ldots$ and $Y$ $S$-valued r.v. with distribution

$$\mathbf{P}\Big(Y_n \in A\Big) = \mu_n\Big(A\Big), \quad \mathbf{P}\Big(Y \in A\Big) = \mu\Big(A\Big), \qquad A \in \mathcal{S}$$

*not necessarily jointly defined.*

**Definition** (Weak convergence of probability measures)**.** $\mu_n \Rightarrow \mu$, *or* $Y_n \Rightarrow Y$, *iff* $\forall f : S \to \mathbb{R}$ *continuous and bounded*

$$\lim_{n\to\infty} \int_S f d\mu_n = \int_S f d\mu, \quad \text{or} \quad \lim_{n\to\infty} \mathbf{E}\Big(f(Y_n)\Big) = \mathbf{E}\Big(f(Y)\Big).$$

**Theorem** (Equiv. characterizations, "portmanteau thm"). .

*(a)≡(b)≡(c)≡(d)*

*(a)* $\mu_n \Rightarrow \mu$.

*(b)* $(\forall A \in \mathcal{S})$, $A$ *open:* $\liminf_{n\to\infty} \mu_n\big(A\big) \geq \mu\big(A\big)$.

*(c)* $(\forall A \in \mathcal{S})$, $A$ *closed:* $\limsup_{n\to\infty} \mu_n\big(A\big) \leq \mu\big(A\big)$.

*(d)* $(\forall A \in \mathcal{S})$, *such that* $\mu\big(\partial A\big) = 0$*:* $\lim_{n\to\infty} \mu_n\big(A\big) = \mu\big(A\big)$.

*Proof. Probability 2.* □

## THE SPECIAL CASE OF $\mathbb{R}$ (OR $\mathbb{R}^d$)

The distribution function helps:

$$F_n(x) := \mathbf{P}\Big(Y_n < x\Big) = \mu_n\Big((-\infty, x)\Big),$$

$$F(x) := \mathbf{P}\Big(Y < x\Big) = \mu\Big((-\infty, x)\Big).$$

**Theorem.** $\mu_n \Rightarrow \mu$ *(also denoted* $F_n \Rightarrow F$*) iff*

$$\lim_{n\to\infty} F_n(x) = F(x), \quad \textit{at all points of continuity of } F.$$

*Proof. Probability 2.* □

**EXAMPLES FOR WEAK CONVERGENCE**

**EX1:** Convergence in probability (*Probability 2, Analysis*)— this is NOT the typical case: $(\Omega, \mathcal{F}, \mathbf{P})$

$Y_n, Y : \Omega \to \mathbb{R}$ defined on the same probab. sp., $\quad Y_n \xrightarrow{\mathbf{P}} Y$

**EX2:** Poisson approximation of binomial (*Probability 1*):

$$Y_n \sim BIN(p_n, n), \quad \lim_{n\to\infty} np_n = \lambda \in (0, \infty), \quad Y \sim POI(\lambda).$$

**EX3:** De Moivre's CLT (*Probability 1*):

$$\widetilde{Y}_n \sim BIN(p, n), \qquad Y_n := \frac{\widetilde{Y}_n - pn}{\sqrt{p(1-p)n}}, \qquad Y \sim N(0, 1).$$

**EX4:** De Moivre's-type CLT for gamma-distributions (*Probability 2*):

$$\widetilde{Y}_n \sim GAM(\lambda, n), \qquad Y_n := \frac{\widetilde{Y}_n - \lambda^{-1} n}{\sqrt{\lambda^{-1} n}}, \qquad Y \sim N(0,1).$$

**EX5:** General CLT for sums of i.i.d. r.v.-s (*Probability 2*) — the typical case:

$$X_n \text{ i.i.d. r.v.-s}, \qquad m := \mathbf{E}\Big( X_j \Big), \qquad \sigma^2 := \mathbf{Var}\Big( X_j \Big),$$

$$Y_n := \frac{\sum_{j=1}^{n} (X_j - m)}{\sigma \sqrt{n}}, \qquad Y \sim N(0,1)$$

## TIGHTNESS

**Definition.** *The sequence of probability measures* $\mu_n$ *on* $(S.\mathcal{S})$*, or the sequence of* $S$*-valuead random variables* $Y_n$*, is tight, if* $(\forall \varepsilon > 0)$ $(\exists K \Subset S)$ *such that*

$$(\forall n): \qquad \mu_n\Big(S \setminus K\Big) < \varepsilon,$$

$$\text{or} \quad \mathbf{P}\Big(Y_n \notin K\Big) < \varepsilon.$$

*In the* $S = \mathbb{R}$ *case* $(\forall \varepsilon > 0)$ $(\exists K < \infty)$ *such that*

$$(\forall n): \qquad \mu_n\Big((-\infty, -K) \cup (K, \infty)\Big) < \varepsilon,$$

$$\text{or} \quad \mathbf{P}\Big(\,|Y_n| > K\,\Big) < \varepsilon.$$

**Proposition.** *If $\mu_n \Rightarrow \mu$ then the sequence $\mu_n$ is tight.*

*Proof.* Easy, if $S$ is locally compact!
Choose

$$\widetilde{K} \Subset K \Subset S \quad \text{s.t.} \quad \mu\Big(S \setminus \widetilde{K}\Big) < \varepsilon/2$$

and

$$f : S \to [0,1] \quad \text{cont., s.t.} \quad f \restriction_{\widetilde{K}} = 0, \quad f \restriction_{S\setminus K} = 1.$$

Then

$$\begin{array}{ccccc} \mu_n\Big(S \setminus K\Big) \le & \displaystyle\int_S f d\mu_n & \le \mu_n\Big(S \setminus \widetilde{K}\Big) & & \\ & \downarrow & & & \\ \mu\Big(S \setminus K\Big) \le & \displaystyle\int_S f d\mu & \le \mu\Big(S \setminus \widetilde{K}\Big) < \varepsilon/2. & & \end{array}$$

Hence, $(\exists n_0 < \infty)$ such that $(\forall n \ge n_0) : \mu_n\Big(S \setminus K\Big) < \varepsilon.$ □

**Theorem** (Helly's theorem). *Let* $\{\mu_n / F_n / Y_n\}$, $n = 1, 2, \ldots$, *be a* *tight* *sequence of* {*probability measures / probability distribution functions / random variables*} *on* $\mathbb{R}$. *Then one can extract a weakly convergent subsequence* $\{\mu_{n_k} / F_{n_k} / Y_{n_k}\}$, $k = 1, 2, \ldots$*:*

$$\{ \quad \mu_{n_k} \Rightarrow \mu \quad / \quad F_{n_k} \Rightarrow F \quad / \quad Y_{n_k} \Rightarrow Y \quad \} \qquad \text{as} \quad k \to \infty.$$

**Theorem** (Prohorov's theorem). *Let* $\{\mu_n \,/\, Y_n\}$ , $n = 1, 2, \ldots$, *be a* *tight* *sequence of* {*probability measures / random variables*} *on the complete separable metric space* $S$. *Then one can extract a weakly convergent subsequence* $\{\mu_{n_k} \,/\, Y_{n_k}\}$ , $k = 1, 2, \ldots$*:*

$$\{ \quad \mu_{n_k} \Rightarrow \mu \quad / \quad Y_{n_k} \Rightarrow Y \quad \} \qquad \text{as} \quad k \to \infty.$$

For proof of both Thms see: *Probability 2.*

## METHODS FOR PROVING WEAK CONVERGENCE

**General scheme:** (1) prove tightness
(2) prove uniqueness of possible limits
(3) identify the limit

**Methods:**

**(A) With bare hands**
(e.g. De Moivre, Poisson, maxima of i.i.d.)

**(B) Method of moments**

**(C) Method of characteristic functions**
(e.g. Markov-Lévy CLT)

**(D) Coupling**

**(E) Mixed methods**

# III.
# WITH BARE HANDS
# (ARCSINE LAWS AND RELATED STUFF)

$X_n$ simple symmetric random walk on $\mathbb{Z}$ ($d = 1!$):

$$X_0 = 0, \qquad \mathbf{P}\Big( X_{n+1} = i \pm 1 \,\Big|\, X_n = i \Big) = \frac{1}{2}.$$

Some relevant random variables:

*The maximum:* $M_n := \max\{X_j : j \in [0, n]\}$,

*First hitting of* $r \in \mathbb{Z}_+$*:* $T_r := \inf\{n > 0 : X_n = r\}$,

*Return times* $k \in \mathbb{N}$*:* $R_0 = 0, \quad R_{k+1} := \inf\{n > R_k : X_n = 0\}$,

*Local time at* $0 \in \mathbb{Z}$*:* $L_n := \#\{j \in (0, n] : X_j = 0\}$,

*Last visit to* $0 \in \mathbb{Z}$*:* $\lambda_n := \max\{j \in (0, n] : X_j = 0\}$,

*Time spent on* $\mathbb{Z}_+$*:* $\pi_n := \#\{j \in (0, n] : \frac{X_{j-1} + X_j}{2} > 0\}$.

**Theorem** (Limit theorem for the maximum)**.**
*(i) Discrete,microscopic version:* $0 \le r \le n$ *fixed:*

$$\mathbf{P}\Big( M_n = r \Big) = \mathbf{P}\Big( X_n = r \Big) + \mathbf{P}\Big( X_n = r + 1 \Big).$$

*(ii) Local limit theorem:* $0 \le u$ *fixed,* $1 \ll n$*:*

$$n^{1/2}\mathbf{P}\Big( M_n = [n^{1/2}u] \Big) = \sqrt{\frac{2}{\pi}} e^{-u^2/2} \mathbb{1}_{u>0} + \mathcal{O}(n^{-1/2})$$

*(iii) Global (integrated) limit theorem:* $0 \le x$ *fixed:*

$$\begin{aligned} \lim_{n\to\infty} \mathbf{P}\Big( n^{-1/2} M_n < x \Big) &= \mathbb{1}_{x>0} \sqrt{\frac{2}{\pi}} \int_0^x e^{-u^2/2} du \\ &= \mathbb{1}_{x>0} (2\Phi(x) - 1). \end{aligned}$$

*Proof of part (i).*

$$
\begin{aligned}
\mathbf{P}\big(M_n \ge r\big) &= \mathbf{P}\big(M_n \ge r, X_n \ne r\big) + \mathbf{P}\big(M_n \ge r, X_n = r\big) \\
&\overset{*}{=} 2\mathbf{P}\big(M_n \ge r, X_n > r\big) + \mathbf{P}\big(M_n \ge r, X_n = r\big) \\
&= 2\mathbf{P}\big(X_n \ge r\big) - \mathbf{P}\big(X_n = r\big).
\end{aligned}
$$

*: due to the reflection principle.

$$
\begin{aligned}
\mathbf{P}\big(M_n = r\big) &= \mathbf{P}\big(M_n \ge r\big) - \mathbf{P}\big(M_n \ge r+1\big) \\
&= 2\mathbf{P}\big(X_n \ge r\big) - 2\mathbf{P}\big(X_n \ge r+1\big) - \\
&\qquad -\mathbf{P}\big(X_n = r\big) + \mathbf{P}\big(X_n = r+1\big) \\
&= \mathbf{P}\big(X_n = r\big) + \mathbf{P}\big(X_n = r+1\big)
\end{aligned}
$$

□

*Proof of parts (ii) and (iii).*

$$\mathbf{P}\Big( M_n = [\sqrt{n}u] \Big) \;=\; \mathbf{P}\Big( X_n = [\sqrt{n}u] \Big) + \mathbf{P}\Big( X_n = [\sqrt{n}u] + 1 \Big)$$

$$\stackrel{**}{=} \; n^{-1/2}\sqrt{\frac{2}{\pi}}e^{-u^2/2} + \mathcal{O}(n^{-1})$$

**: due to De Moivre.

(iii) Integrated version folows from local version + Fatou + Riemannian integration.

□

**Theorem** (Limit theorem for the hitting times)**.**
*(i) Discrete, microscopic version:* $0 < r \le n$ *fixed:*

$$\mathbf{P}\Big(T_r = n\Big) = \frac{r}{n}\binom{n}{(n+r)/2}2^{-n}$$

*(ii) Local limit theorem:* $0 < s$ *fixed,* $1 \ll r$*:*

$$r^2\mathbf{P}\Big(T_r = [r^2 s]\Big) = \sqrt{\frac{2}{\pi}}s^{-3/2}e^{-1/(2s)}1\!\!1_{s>0} + \mathcal{O}(r^{-1}).$$

*(iii) Global (integrated) limit theorem:* $0 < t$ *fixed:*

$$\begin{aligned} \lim_{r\to\infty}\mathbf{P}\Big(r^{-2}T_r < t\Big) &= 1\!\!1_{t>0}\frac{1}{\sqrt{2\pi}}\int_0^t s^{-3/2}e^{-1/(2s)}ds \\ &= 1\!\!1_{t>0}\sqrt{\frac{2}{\pi}}\int_{1/\sqrt{t}}^{\infty} e^{-u^2/2}du. \end{aligned}$$

*Proof of part (i).*

$$\mathbf{P}\Big(T_r = n\Big) = \frac{1}{2}\mathbf{P}\Big(\{\max_{j\le n-2} X_j \le r-1\} \wedge \{X_{n-1} = r-1\}\Big)$$

$$= \frac{1}{2}\mathbf{P}\Big(X_{n-1} = r-1\Big) -$$

$$-\frac{1}{2}\mathbf{P}\Big(\{\max_{j\le n-2} X_j \ge r\} \wedge \{X_{n-1} = r-1\}\Big)$$

$$\overset{*}{=} \frac{1}{2}\mathbf{P}\Big(X_{n-1} = r-1\Big) - \frac{1}{2}\mathbf{P}\Big(X_{n-1} = r+1\Big)$$

$$= \frac{r}{n}\binom{n}{(n+r)/2}2^{-n}$$

*: due to the reflection principle. □

*Proof of parts (ii) and (iii).*

$$\mathbf{P}\Big(T_r = [r^2 s]\Big) = \frac{r}{[r^2 s]}\binom{[r^2 s]}{([r^2 s]+r)/2} 2^{-[r^2 s]}$$

$$\stackrel{**}{=} r^{-2}\frac{2}{\sqrt{2\pi}} s^{-3/2} e^{-1/(2s)} + \mathcal{O}(r^{-3})$$

**: due to Stirling.

(iii) Integrated version: local version + Fatou + Riemannian integration. □

**Theorem** (Limit theorem for the return times).
*(i) Discrete, microscopic version:* $0 < k \leq n$ *fixed:*

$$\mathbf{P}\Big( R_k = k + n \Big) = \frac{k}{n} \binom{n}{(n+k)/2} 2^{-n}$$

*(ii) Local limit theorem:* $0 < s$ *fixed:*

$$k^2 \mathbf{P}\Big( R_k = [k^2 s] \Big) = \frac{1}{\sqrt{2\pi}} s^{-3/2} e^{-1/(2s)} \mathbb{1}_{s>0} + \mathcal{O}(k^{-1}).$$

*(ii) Global (integrated) version:* $0 < t$ *fixed:*

$$\begin{aligned} \lim_{k\to\infty} \mathbf{P}\Big( k^{-2} R_k < t \Big) &= \mathbb{1}_{t>0} \frac{1}{\sqrt{2\pi}} \int_0^t s^{-3/2} e^{-1/(2s)} ds. \\ &= \mathbb{1}_{t>0} \sqrt{\frac{2}{\pi}} \int_{1/\sqrt{t}}^{\infty} e^{-u^2/2} du. \end{aligned}$$

*Proof.*

$$R_k \overset{\text{law}}{=} T_k + k.$$

□

**Remarks on the last two limit theorems:**

**(1) I.i.d. sums:**

$$T_r = \xi_1 + \xi_2 + \cdots + \xi_r, \quad R_k = \zeta_1 + \zeta_2 + \cdots + \zeta_k,$$

where $\xi_i$, $i = 1, 2, \ldots$ and $\zeta_i$, $i = 1, 2, \ldots$ are sequences of i.i.d. r.v.-s with

$$\xi_i \overset{\text{law}}{=} T_1, \qquad \zeta_i \overset{\text{law}}{=} R_1 \overset{\text{law}}{=} T_1 + 1.$$

**(2) Stability:**

$$f_1(s) := \frac{1}{\sqrt{2\pi}} s^{-3/2} e^{-1/(2s)} 1\!\!1_{s>0}, \qquad f_a(s) := a f_1(as), \quad a > 0.$$

Then

$$f_a * f_b = f_{(\sqrt{a}+\sqrt{b})^2}$$

Homework.

**Theorem** (Limit theorem for the local time at zero.)**.**
*Global (integrated) version:*

$$\lim_{n\to\infty} \mathbf{P}\Big( n^{-1/2} L_n < t \Big) = 1\!\!1_{t>0} \sqrt{\frac{2}{\pi}} \int_0^t e^{-u^2/2} du.$$

*Proof.*

$$\{L_n < k\} = \{R_k > n\}.$$

Hence

$$\begin{aligned} \lim_{n\to\infty} \mathbf{P}\Big( L_n < n^{1/2} t \Big) &= \lim_{n\to\infty} \mathbf{P}\Big( R_{n^{1/2}t} > n \Big) = \lim_{m\to\infty} \mathbf{P}\Big( R_m > m^2/t^2 \Big) \\ &= \sqrt{\frac{2}{\pi}} \int_0^t e^{-u^2/2} du. \end{aligned}$$

□

**Remark:** Note that

$$\lim_{n\to\infty} \mathbf{P}\Big(n^{-1/2}|X_n| < u\Big) = \lim_{n\to\infty} \mathbf{P}\Big(n^{-1/2}L_n < u\Big)$$

$$= \lim_{n\to\infty} \mathbf{P}\Big(n^{-1/2}M_n < u\Big)$$

For a simple symmetric random walk $X_n$ (on $\mathbb{Z}$) denote

$$u(n) := \mathbf{P}\Big( X_n = 0 \Big) = \binom{n}{n/2} 2^{-n}$$

$$f(n) := \mathbf{P}\Big( \min\{m \geq 1 : X_m = X_0\} = n \Big)$$

Recall the identity:

$$u(n) = \sum_{m=0}^{n} f(m) u(n-m).$$

**Theorem** (Paul Lévy's arcsine theorem).
*(i) Discrete, microscopic version:* $0 \le k \le n$:

$$\mathbf{P}\Big(\lambda_{2n+1} = 2k\Big) \stackrel{\checkmark}{=} \mathbf{P}\Big(\lambda_{2n} = 2k\Big) = u(2k)u(2n-2k),$$

$$\mathbf{P}\Big(\pi_{2n+1} \in \{2k, 2k+1\}\Big) \stackrel{\checkmark}{=} \mathbf{P}\Big(\pi_{2n} = 2k\Big) = u(2k)u(2n-2k),$$

$$\bigg(\mathbf{P}\Big(\lambda_{2n} = 2k+1\Big) \stackrel{\checkmark}{=} \mathbf{P}\Big(\lambda_{2n+1} = 2k+1\Big) \stackrel{\checkmark}{=} \mathbf{P}\Big(\pi_{2n} = 2k+1\Big) \stackrel{\checkmark}{=} 0\bigg)$$

*(ii) Local limit theorem:* $y \in (0,1)$ *fixed* $1 \ll n$:

$$n\mathbf{P}\Big(\lambda_{2n} = 2[ny]\Big) = n\mathbf{P}\Big(\pi_{2n} = 2[ny]\Big) = \frac{1}{\pi}\frac{1}{\sqrt{y(1-y)}} + \mathcal{O}(n^{-1/2})$$

*(ii) Global (integrated) limit theorem:* $x \in (0,1)$ *fixed*

$$\lim_{n\to\infty} \mathbf{P}\Big(n^{-1}\lambda_n < x\Big) = \lim_{n\to\infty} \mathbf{P}\Big(n^{-1}\pi_n < x\Big) = 1\!\!1_{0<x<1}\frac{2}{\pi}\arcsin\sqrt{x}.$$

**Lemma.**

$$\mathbf{P}\Big( X_j \neq 0,\ j = 1, 2, \ldots, 2n \Big) = \mathbf{P}\Big( X_{2n} = 0 \Big) =: u(2n).$$

*Proof of the Lemma.*

$$\mathbf{P}\Big( X_j \neq 0,\ j = 1, 2, \ldots, 2n \Big) = 2\mathbf{P}\Big( X_j > 0,\ j = 1, 2, \ldots, 2n \Big)$$

$$= 2 \sum_{r=1}^{\infty} \mathbf{P}\Big( \{X_j > 0,\ j = 1, 2, \ldots, 2n-1\} \wedge \{X_{2n} = 2r\} \Big)$$

$$\overset{*}{=} 2 \sum_{r=1}^{\infty} \frac{1}{2}\Big(\mathbf{P}\Big( X_{2n-1} = 2r - 1 \Big) - \mathbf{P}\Big( X_{2n-1} = 2r + 1 \Big)\Big)$$

$$= \mathbf{P}\Big( X_{2n-1} = 1 \Big) = \mathbf{P}\Big( X_{2n} = 0 \Big).$$

*: due to the reflection principle. □

*Proof of the Theorem*

(i) For $\lambda_n$:

$$\begin{aligned}
\mathbf{P}\Big(\lambda_{2n} = 2k\Big) &= \mathbf{P}\Big(\{X_{2k=0}\} \wedge \{X_j \neq 0,\ j = 2k+1, \ldots, 2n\}\Big) \\
&= \mathbf{P}\Big(X_{2k=0}\Big)\mathbf{P}\Big(X_j \neq 0,\ j = 1, \ldots, 2n-2k)\}\Big) \\
&= u(2k)u(2n-2k).
\end{aligned}$$

For $\pi_n$ by induction. Note that

$$\mathbf{P}\Big(\pi_{2n} = 2k\Big) = \mathbf{P}\Big(\pi_{2n} = 2n - 2k\Big).$$

For $k = 0$ or $k = n$:

$$\begin{aligned}
\mathbf{P}\Big(\pi_{2n} = 0\Big) &= \mathbf{P}\Big(X_j \geq 0,\ j = 1, 2, \ldots, 2n\Big) \\
&= \mathbf{P}\Big(X_j \geq 0,\ j = 1, 2, \ldots, 2n-1\Big) \\
&= 2\mathbf{P}\Big(X_j > 0,\ j = 1, 2, \ldots, 2n\Big) = u(2n)u(0)
\end{aligned}$$

Denote

$$b(2n, 2k) := \mathbf{P}\Big(\pi_{2n} = 2k\Big) = b(2n, 2n - 2k)$$

For $1 \leq k \leq n$ there is a first excursion to the left or to the right:

$$b(2n, 2k) = \frac{1}{2}\sum_{r=1}^{k} f(2r) b(2n-2r, 2k-2r) + \frac{1}{2}\sum_{r=1}^{n-k} f(2r) b(2n-2r, 2k)$$

By the induction assumption:

$$\begin{aligned} b(2n, 2k) &= \frac{1}{2}u(2n - 2k)\sum_{r=1}^{k} f(2r)u(2k - 2r) + \\ &\qquad + \frac{1}{2}u(2k)\sum_{r=1}^{n-k} f(2r)u(2n - 2k - 2r) \\ &= \frac{1}{2}u(2n - 2k)u(2k) + \frac{1}{2}u(2k)u(2n - 2k) = u(2k)u(2n - 2k) \end{aligned}$$

(ii)

$$u(2[ny])u(2[n(1-y)]) \overset{**}{=} n^{-1}\frac{1}{\pi}\frac{1}{\sqrt{y(1-y)}} + \mathcal{O}(n^{-3/2})$$

**: due to Stirling.

(iii) Integrated version: local version + Fatou + Riemannian integration.

$\square$

# IV.
# THE METHOD OF MOMENTS AND THE METHOD OF CHARACTERISTIC FUNCTIONS

## RECALL EVERYTHING YOU LEARNT ABOUT CHARACTERISTIC FUNCTIONS

## PROBABILITY II.

## THE METHOD OF MOMENTS

Let $X$ be a random variable, its absolute moments and its moments are assumed finite:

$$A_k := \mathbf{E}\Big( |X|^k \Big) < \infty, \qquad M_k := \mathbf{E}\Big( X^k \Big)$$

**Remark:** In order that the sequences $A_k$ and $M_k$ be the sequences of (absolute) moments of a random variable $X$ it must satisfy an infinite set of (Jensen-type) inequalities: in particular, if $k_1 + \cdots + k_m = k$, respectively, if $k_1 + \cdots + k_m = 2k$ then

$$\prod_{j=1}^{m} A_{k_j} \leq A_k, \qquad \prod_{j=1}^{m} |M_{k_j}| \leq M_{2k},$$

**The "Moment problem":** Given a sequence of moments $M_k$, does it determine uniquely the distribution of a random variable?

**Theorem.** *If $M_k$ is a sequence of moments such that*

$$\limsup_{k\to\infty} \left(\frac{|M_k|}{k!}\right)^{1/k} := R^{-1} < \infty$$

*then it determines a unique random variable $X$ (or: probability distribution) such that $M_k = \mathbf{E}\left( X^k \right)$.*

*Proof.* The power series of the characteristic function

$$\sum_{k=0}^{\infty} \frac{M_k}{k!} (iu)^k$$

will have radius of convergence $R > 0$, and thus it will be uniquely determined. □

**Examples:** Compute all moments of all remarkable distributions. Eg.

$$X \sim EXP(\lambda): \qquad M_k = A_k = \lambda^{-k} k!$$

$$X \sim N(0,\sigma): \qquad A_{2k} = \sigma^{2k} \frac{2k!}{2^k k!} = M_{2k},$$

$$A_{2k+1} = \sigma^{2k+1} \sqrt{\frac{2}{\pi}} 2^k k!, \quad M_{2k+1} = 0$$

**A counterexample:** the *log-normal* distribution (HW!).

**Weak limit from convergence of moments:**

**Theorem.** *Let* $Z_n$ *be a sequence of random variables which have all moments finite and denote*

$$M_{n,k} := \mathbf{E}\Big( Z_n^k \Big).$$

*If* $(\forall k)$ *the limit* $\lim_{n\to\infty} M_{n,k} =: M_k$ *exists and the sequence of moments* $M_k$ *determines* uniquely *a distribution/random variable* $Z$, *then* $Z_n \Rightarrow Z$.

**Remark:** The sequence $M_k$ is a sequence of moments.

*Proof* (1) Tightness:

$$\mathbf{P}\Big( |Z_n| > K \Big) \leq \frac{M_{n,2}}{K^2}. \leq \frac{\sup_n M_{n,2}}{K^2}.$$

(2) Identification of the limit: Assume $Z_{n'} \Rightarrow \widetilde{Z}$. For $K < \infty$ let $\varphi_K : \mathbb{R} \to \mathbb{R}$,

$$\varphi_K(x) := x\,1\!\!1_{|x|\le K} + \mathrm{sgn}(x) K\, 1\!\!1_{|x|>K}.$$

Then

$$\begin{aligned}
\mathbf{E}\Big(\widetilde{Z}^k\Big) &= \lim_{K\to\infty} \mathbf{E}\Big(\varphi_K(\widetilde{Z})^k\Big) \\
&= \lim_{K\to\infty} \lim_{n'\to\infty} \mathbf{E}\Big(\varphi_K(Z_{n'})^k\Big) \qquad \text{(due to weak cvg.)} \\
&= \lim_{K\to\infty} \lim_{n'\to\infty} \Big(\mathbf{E}\Big(Z_{n'}^k\Big) - \mathbf{E}\Big(Z_{n'}^k - \varphi_K(Z_{n'})^k\Big)\Big) \\
&= \lim_{n'\to\infty} M_{n',k} - \lim_{K\to\infty} \lim_{n'\to\infty} \mathbf{E}\Big(Z_{n'}^k - \varphi_K(Z_{n'})^k\Big)
\end{aligned}$$

But:

$$\left|\mathbf{E}\Big( Z_{n'}^k - \varphi_K(Z_{n'})^k \Big)\right| \le \mathbf{E}\Big( \big|Z_{n'}\big|^k \mathbb{1}_{|Z_n'|>K} \Big)$$

$$\overset{(1)}{\le} \sqrt{M_{n',2k}}\,\sqrt{\mathbf{P}\Big(|Z_n'| > K\Big)}$$

$$\overset{(2)}{\le} \frac{\sqrt{M_{n',2k}}\,\sqrt{M_{n',2}}}{K}$$

(1): due to Schwarz's inequality
(2): due to Markov's inequality

Altogether:

$$\mathbf{E}\Big(\widetilde{Z}^k\Big) = M_k.$$

□

**Appl 1: CLT with the method of moments:** Sheds light on the *combinatorial aspects* of the CLT. Let $\xi_j$ be i.i.d. with all moments finite, $\mathbf{E}\big(\xi_j^k\big) =: m_k, \quad m_1 = 0, \quad m_2 =: \sigma^2$,

$$Z_n := \frac{\xi_1 + \cdots + \xi_n}{\sqrt{n}}.$$

Then, with fixed $k$:

$$\begin{aligned} \mathbf{E}\big(Z_n^{2k}\big) &= n^{-k}\binom{n}{k}\sigma^{2k}\frac{2k!}{2^k} + o(1) \to \sigma^{2k}\frac{2k!}{2^k k!}, \\ \mathbf{E}\big(Z_n^{2k+1}\big) &= o(1) \to 0, \end{aligned}$$

as $n \to \infty$ (with $k$ fixed).

**THE METHOD OF CHARACTERISTIC FUNCTIONS (Repeat from Probability II.)**

**Theorem.** *Let* $Z_n$ *be a sequence of random variables and* $\varphi_n : \mathbb{R} \longrightarrow \mathbb{R}$ *their characteristic functions,*

$$\varphi_n(u) := \mathbf{E}\Big( \exp(iuZ_n) \Big).$$

*If*

$$(\forall u \in \mathbb{R}) : \quad \lim_{n \to \infty} \varphi_n(u) = \varphi(u) \qquad \textit{(pointwise!)}$$

*and* $u \mapsto \varphi(u)$ *is continuous at* $u = 0$*, then* $\varphi$ *is characteristic function of a random variable* $Z$ *and* $Z_n \Rightarrow Z$*.*

*Proof:*
(1) Tightness:

**Lemma** (Paul Lévy). *Let* $Y$ *be a random variable and* $\psi(u) := \mathbf{E}\Big( \exp(iuY) \Big)$ *its characteristic function. Then for any* $K < \infty$

$$\mathbf{P}\Big( |Y| > K \Big) \leq \frac{K}{2} \int_{-2/K}^{2/K} (1 - \psi(u))du.$$

*Proof of the Lemma:*

$$\begin{aligned}
\frac{K}{2}\int_{-2/K}^{2/K}(1-\psi(u))du &= \frac{K}{2}\int_{-2/K}^{2/K}\mathbf{E}\Big(1-e^{iuY}\Big)du \\
&\overset{(1)}{=} 2\mathbf{E}\Big(1-\frac{\sin(2Y/K)}{2Y/K}\Big) \\
&\overset{(2)}{\geq} 2\mathbf{E}\Big(\Big(1-\frac{\sin(2Y/K)}{2Y/K}\Big)\mathbb{1}_{|Y|>K}\Big) \\
&\overset{(3)}{\geq} 2\mathbf{E}\Big(\Big(1-\frac{K}{2|Y|}\Big)\mathbb{1}_{|Y|>K}\Big) \\
&\geq \mathbf{P}\Big(|Y|>K\Big).
\end{aligned}$$

(1): Fubini,
(2): $|\sin\alpha/\alpha| \leq 1$,
(3): $\sin\alpha/\alpha \leq 1/|\alpha|$.

□

*Proof of the Theorem continued:*
From continuity of $u \mapsto \varphi(u)$ at $u = 0$:

$$(\exists K < \infty): \qquad \frac{K}{2}\int_{-2/K}^{2/K}(1-\varphi(u))du < \frac{\varepsilon}{2}.$$

From pointwise convergence (and uniform boundedness of $\varphi_n$)

$$(\exists n_0 < \infty): \qquad (\forall n \geq n_0): \quad \frac{K}{2}\int_{-2/K}^{2/K}(1-\varphi_n(u))du < \varepsilon.$$

Hence tightness, by the Lemma.

(2) Identification of the limit: Assume $Z_{n'} \Rightarrow \widetilde{Z}$, then

$$\mathbf{E}\Big(\exp(iu\widetilde{Z})\Big) = \lim_{n'\to\infty}\mathbf{E}\Big(\exp(iuZ_{n'})\Big) = \varphi(u).$$

□

**ERDŐS-KAC THEOREM: CLT FOR NUMBER OF PRIME DIVISORS** (a mixture of the method of characteristic functions and method of moments)

Denote by $\mathbb{P}$ the set of primes and

$$g : \mathbb{N} \to \mathbb{N}, \qquad g(m) := \#\{p \in \mathbb{P} : p \,|\, m\}.$$

**Theorem** (Paul Erdős & Marc Kac, 1940)**.**

$$\lim_{n\to\infty} n^{-1} \# \Big\{ m \in \{1, 2, \ldots, n\} : \frac{g(m) - \log\log n}{\sqrt{\log\log n}} < x \Big\} = \int_{-\infty}^{x} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy.$$

**Probabilistic setup:** Let $\omega_n$ be randomly sampled from $(\{1, 2, \ldots, n\}, UNI)$ and $Z_n := g(\omega_n)$. Then

$$\frac{Z_n - \log\log n}{\sqrt{\log\log n}} \Rightarrow N(0, 1).$$

*Proof:*
We will use

$$\sum_{p \in \mathbb{P}: p \leq n} \frac{1}{p} = \log \log n + \mathcal{O}(1).$$

Define the random variables $Y_{n,p}$, $p \in \mathbb{P}$, $n \in \mathbb{N}$.

$$Y_{n,p} := 1\!\!1_{p \,|\, \omega_n}, \quad \text{where} \quad \omega_n \sim UNI(\{1, 2, \ldots, n\}).$$

Mind that for $n \in \mathbb{N}$ fixed $\left(Y_{n,p}\right)_{p \in \mathbb{P}}$ are jointly defined.
Then

$$Z_n = \sum_{p \in \mathbb{P}} Y_{n,p}.$$

Note that for any $k < \infty$ and $p_1, p_2, \ldots, p_k \in \mathbb{P}$ fixed

$$\left(Y_{n,p_1}, Y_{n,p_2}, \ldots, Y_{n,p_k}\right) \Rightarrow \left(X_{p_1}, X_{p_2}, \ldots, X_{p_k}\right)$$

where $X_p$, $p \in \mathbb{P}$, are (jointly defined) *independent* random variables with distribution

$$\mathbf{P}\Big( X_p = 1 \Big) = \frac{1}{p} = 1 - \mathbf{P}\Big( X_p = 0 \Big).$$

How to guess the result? Let

$$\alpha_n \to \infty, \qquad S_n := \sum_{p \in \mathbb{P} : p \le \alpha_n} X_p.$$

Then

$$S_n^* := \frac{S_n - \log\log \alpha_n}{\sqrt{\log\log \alpha_n}} \Rightarrow N(0,1).$$

Note that

$$\frac{S_n - \log\log \alpha_n}{\sqrt{\log\log \alpha_n}} = \frac{S_n - \mathbf{E}\Big( S_n \Big)}{\sqrt{\log\log \alpha_n}} + \frac{\mathbf{E}\Big( S_n \Big) - \log\log \alpha_n}{\sqrt{\log\log \alpha_n}}$$

and

$$\frac{\mathbf{E}\Big( S_n \Big) - \log\log\alpha_n}{\sqrt{\log\log\alpha_n}} = \frac{\log\log\log\alpha_n + \mathcal{O}(1)}{\sqrt{\log\log\alpha_n}} \to 0$$

The weak convergence

$$\frac{S_n - \mathbf{E}\Big( S_n \Big)}{\sqrt{\log\log\alpha_n}} \Rightarrow N(0,1)$$

is proved with method of characteristic functions:

$$\mathbf{E}\Big( \exp(iuS_n^*) \Big) = \prod_{p\in\mathbb{P}:p\le\alpha_n} \left( \frac{1}{p} \exp\{\frac{iu(p-1)/p}{\sqrt{\log\log\alpha_n}}\} + \frac{p-1}{p} \exp\{\frac{-iu/p}{\sqrt{\log\log\alpha_n}}\} \right)$$

$$\to \exp\{-u^2/2\}$$

HW!

Let:

$$\begin{aligned} \alpha_n &:= n^{1/\log\log n} \\ \log \alpha_n &= \frac{\log n}{\log\log n} \\ \log\log \alpha_n &= \log\log n - \log\log\log n. \end{aligned}$$

Note that

(1): $$(\forall \varepsilon > 0) : \alpha_n = o(n^{\varepsilon}),$$

(2): $$\sum_{\alpha_n < p \leq n} \frac{1}{p} = \log\log\log n + \mathcal{O}(1).$$

Let

$$S_n := \sum_{p\in\mathbb{P}:p\le\alpha_n} X_p, \qquad S_n^* := \frac{S_n - \log\log\alpha_n}{\sqrt{\log\log\alpha_n}}$$

$$T_n := \sum_{p\in\mathbb{P}:p\le\alpha_n} Y_{n,p}, \qquad T_n^* := \frac{T_n - \log\log\alpha_n}{\sqrt{\log\log\alpha_n}}$$

$$Z_n := \sum_{p\in\mathbb{P}:p\le n} Y_{n,p} = \sum_{p\in\mathbb{P}} Y_{n,p}, \qquad Z_n^* := \frac{Z_n - \log\log n}{\sqrt{\log\log n}}$$

We know that $S_n^* \Rightarrow N(0,1)$ and we want to prove $Z_n^* \Rightarrow N(0,1)$.

Step 1.

$$\begin{aligned}
\mathbf{E}\Big(\,|Z_n - T_n|\,\Big) &= \sum_{p\in\mathbb{P}:\alpha_n<p\le n} \mathbf{E}\Big(Y_{n,p}\Big) \le \sum_{p\in\mathbb{P}:\alpha_n<p\le n} \frac{1}{p} \\
&= \log\log\log n + \mathcal{O}(1) = o(\sqrt{\log\log n})
\end{aligned}$$

$$|\log\log n - \log\log \alpha_n| = \log\log\log n + \mathcal{O}(1) = o(\sqrt{\log\log n})$$

Hence

$$|T_n^* - Z_n^*| \xrightarrow{\mathbf{P}} 0.$$

Step 2. We prove $T_n^* \Rightarrow N(0,1)$ with method of moments.

By computation:

$$\lim_{n\to\infty} \mathbf{E}\left( S_n^k \right) = \int_{-\infty}^{\infty} \frac{e^{-y^2/2}}{\sqrt{2\pi}} y^k dy =: M_k. \qquad \text{HW!}$$

For $1 < p_1 < p_2 < \cdots < p_l \leq \alpha_n$ and $k_1, k_2, \ldots, k_l \geq 1$:

$$\mathbf{E}\left( X_{p_1}^{k_1} X_{p_2}^{k_2} \ldots X_{p_l}^{k_l} \right) = \mathbf{E}\left( X_{p_1} X_{p_2} \ldots X_{p_l} \right) = \frac{1}{p_1 p_2 \cdots p_l}$$

$$\mathbf{E}\left( Y_{n,p_1}^{k_1} Y_{n,p_2}^{k_2} \ldots Y_{n,p_l}^{k_l} \right) = \mathbf{E}\left( Y_{n,p_1} Y_{n,p_2} \ldots Y_{n,p_l} \right) = \frac{1}{n} \left\lfloor \frac{n}{p_1 p_2 \ldots p_l} \right\rfloor .$$

Hence:

$$\left|\mathbf{E}\Big(X_{p_1}^{k_1}X_{p_2}^{k_2}\dots X_{p_l}^{k_l}\Big)-\mathbf{E}\Big(Y_{n,p_1}^{k_1}Y_{n,p_2}^{k_2}\dots Y_{n,p_l}^{k_l}\Big)\right|\le\frac{1}{n}.$$

Using this and

$$(x_1+x_2+\cdots+x_N)^k=$$

$$=\sum_{l=1}^{N}\sum_{\substack{k_1,k_2,\dots,k_l\ge 1\\ k_1+k_2+\cdots+k_l=k}}\sum_{1\le m_1<m_2<\cdots<m_l\le N}C(l;k_1,k_2,\dots,k_l)x_{m_1}^{k_1}x_{m_2}^{k_2}\dots x_{m_l}^{k_l}$$

we readily obtain

$$\left|\mathbf{E}\Big(S_n^k\Big)-\mathbf{E}\Big(T_n^k\Big)\right|\le\frac{\alpha_n^k}{n}=o(1)$$

and thus

$$\lim_{n \to \infty} \mathbf{E}\left( T_n^k \right) = M_k.$$

Hence:

$$T_n^* \Rightarrow N(0, 1),$$

which together with "Step 1" implies

$$Z_n^* \Rightarrow N(0, 1),$$

□

**LIMIT THEOREM FOR THE COUPON COLLECTOR**
(mixture of "bare hands" and characteristic/generating function method)

For $n \in \mathbb{N}$, let $\xi_{n,k}$, $k = 0, 1, \ldots, n-1$ be independent geometrically distributed random variables with distribution

$$\mathbf{P}\big(\xi_{n,k} = m\big) = \Big(\frac{k}{n}\Big)^m \frac{n-k}{n}, \qquad m = 0, 1, 2, \ldots$$

and

$$V_n := \sum_{k=0}^{n-1} \xi_{n,k}$$

Then

$$\mathbf{E}\left(\xi_{n,k}\right) = \frac{k}{n-k}, \qquad \mathbf{Var}\left(\xi_{n,k}\right) = \frac{nk}{(n-k)^2}$$

$$\mathbf{E}\left(V_n\right) = n\log n + \mathcal{O}(n), \qquad \mathbf{Var}\left(V_n\right) = \frac{\pi^2}{6}n^2 + \mathcal{O}(n\log n).$$

**Theorem.**

$$\lim_{n\to\infty} \mathbf{P}\Big(\frac{V_n - n\log n}{n} < x\Big) = \exp\{-e^{-x}\}.$$

**Remark:** The (two-parameter family of) distributions

$$\begin{aligned} F_{a,b}(x) &:= \exp\{-e^{-ax+b}\}, \qquad a \in \mathbb{R}_+, b \in \mathbb{R}, \\ f_{a,b}(x) &:= \frac{d}{dx}F_{a,b}(x) = a\exp\{-e^{-ax+b} - ax + b\} \end{aligned}$$

are called Type-1 Gumbel distributions and appear in extreme value theory.

*Proof:* Let $\zeta_{n,k} := \xi_{n,n-k}$, $k = 1, \ldots, n$, and

$$Z_n := \sum_{k=1}^{n} \left( \frac{\zeta_{n,k}}{n} - \frac{1}{k} \right) = \frac{V_n - n \log n}{n} - \gamma + \mathcal{O}(n^{-1}).$$

where $\gamma$ is Euler's constant

$$\gamma := \lim_{n \to \infty} \left( \sum_{k=1}^{n} k^{-1} - \log n \right) \approx 0.5772\ldots.$$

**Lemma.** *Let $p_n \searrow 0$ so that $np_n \to \lambda \in \mathbb{R}_+$ and $\zeta_n$ be a sequence of geometrically distributed random variables with distribution*

$$\mathbf{P}\Big( \zeta_n = r \Big) = (1 - p_n)^r p_n.$$

*Then $\zeta_n / n \Rightarrow EXP(\lambda)$.*

*Proof.* Straightforward elementary computation. □

Thus

$$\left(\frac{\zeta_{n,1}}{n}, \frac{\zeta_{n,2}}{n}, \dots\right) \Rightarrow (\zeta_1, \zeta_2, \dots)$$

where $\zeta_k$, $k = 1, 2, \dots$ are independent $EXP(k)$-distributed,

$$\mathbf{E}\big(\zeta_k\big) = \frac{1}{k}, \qquad \mathbf{Var}\big(\zeta_k\big) = \frac{1}{k^2}, \qquad \widetilde{\zeta}_k := \zeta_k - \mathbf{E}\big(\zeta_k\big).$$

It follows that

$$Z_n \Rightarrow Z := \lim_{K\to\infty} \sum_{k=1}^{K} \widetilde{\zeta}_k$$

Note that the limit defining $Z$ exists a.s. due to Kolmogorov's inequality (see Probability II.)

Computing the distribution of $Z$: Let $\Phi : (-1, \infty) \to \mathbb{R}_+$ be the moment generating function (Laplace transform) of $Z$:

$$\Phi(u) \;:=\; \mathbf{E}\Big(\exp(-uZ)\Big) = \prod_{k=1}^{\infty} \mathbf{E}\Big(\exp(-u\widetilde{\zeta}_k)\Big) = \cdots$$

$$= \;\exp \sum_{k=1}^{\infty} \Big(\log \frac{k}{k+u} + \frac{u}{k}\Big)$$

(Mind that the sum is absolutely convergent!)

Analiticity of $(-1, \infty) \ni u \mapsto \Phi(u)$ and the identities

$$\Phi(0) = 1, \qquad \Phi(u+1) = e^{\gamma}(u+1)\Phi(u) \qquad HW!$$

determine

$$\Phi(u) = e^{\gamma u}\Gamma(u+1).$$

On the other hand:

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-uy} d\,\mathsf{exp}\{-e^{-(y+\gamma)}\} &= \int_{-\infty}^{\infty} e^{-uy}\,\mathsf{exp}\{-e^{-(y+\gamma)}\} e^{-(y+\gamma)} dy \\ &= e^{\gamma u} \int_0^{\infty} z^u e^{-z} dz \\ &= e^{\gamma u}\Gamma(u+1).\end{aligned}$$

□

# V.
# LINDEBERG'S THEOREM
# AND ITS APPLICATIONS

## TRIANGULAR ARRAYS OF RANDOM VARIABLES:

Let $N_n \in \mathbb{N}$, $\lim_{n\to\infty} N_n = \infty$ and

$$\xi_{n,k}, \quad k = 1, 2, \ldots, N_n, \quad n = 1, 2, \ldots$$

random variables. Explicitly:

$$\begin{array}{llllllll}
\xi_{1,1}, & \ldots, & \xi_{1,N_1} & & & & & \\
\xi_{2,1}, & \xi_{2,2}, & \ldots, & \xi_{2,N_2} & & & & \\
\xi_{3,1}, & \xi_{3,2}, & \xi_{3,3}, & \ldots, & \xi_{3,N_3} & & & \\
\ldots, & \ldots, & \ldots, & \ldots, & \ldots & \ldots & & \\
\xi_{n,1}, & \xi_{n,2}, & \xi_{n,3}, & \xi_{n,4}, & \xi_{n,5}, & \ldots, & \xi_{n,N_n} & \\
\ldots, & \ldots, & \ldots, & \ldots, & \ldots & \ldots & \ldots & \ldots
\end{array}$$

which are row-wise independent. (Different rows are not even jointly defined.)

Assume:

$$\mathbf{E}\Big( \xi_{n,k} \Big) = 0, \qquad \mathbf{Var}\Big( \xi_{n,k} \Big) =: \sigma^2_{n,k} < \infty$$

and denote their characteristic functions

$$\varphi_{n,k}(u) := \mathbf{E}\Big( \exp\{iu\xi_{n,k}\} \Big).$$

Let

$$S_n := \xi_{n,1} + \xi_{n,2} + \cdots + \xi_{n,N_n}.$$

Then

$$\mathbf{E}\Big( S_n \Big) = 0,$$

$$\mathbf{Var}\Big( S_n \Big) = \sigma^2_{n,1} + \sigma^2_{n,2} + \cdots + \sigma^2_{n,N_n} =: \sigma^2_n$$

Question: CLT for $\dfrac{S_n}{\sigma_n}$?

**Theorem** (Lindeberg, 1922). *If* $(\forall \varepsilon > 0)$

$$\lim_{n\to\infty} \frac{1}{\sigma_n^2} \sum_{k=1}^{N_n} \mathbf{E}\Big( |\xi_{n,k}|^2 \mathbb{1}_{|\xi_{n,k}|>\varepsilon\sigma_n} \Big) = 0 \qquad (***)$$

*then*

$$\frac{S_n}{\sigma_n} \Rightarrow N(0,1).$$

**Comments, remarks:**

(1) Condition (***) is Lindeberg's condition.

(2) W.l.o.g we may assume $(\forall n) : \sigma_n = 1$.

(3) The "meaning" of Lindeberg's condition:
"All components $\xi_{n,k}$ are negligibly tiny compared with $S_n$."
In particular it follows that

$$\lim_{n\to\infty} \max_{1\le k\le N_n} \frac{\sigma_{n,k}^2}{\sigma_n^2} = 0. \qquad (*)$$

Indeed,

$$\begin{aligned}\sigma_{n,k}^2 &= \mathbf{E}\Big(\xi_{n,k}^2 \mathbb{1}_{|\xi_{n,k}|\le\varepsilon\sigma_n}\Big) + \mathbf{E}\Big(\xi_{n,k}^2 \mathbb{1}_{|\xi_{n,k}|>\varepsilon\sigma_n}\Big)\\ &\le \varepsilon^2\sigma_n^2 + \mathbf{E}\Big(\xi_{n,k}^2 \mathbb{1}_{|\xi_{n,k}|>\varepsilon\sigma_n}\Big).\end{aligned}$$

BUT: condition (*) is genuinely weaker then (***) and it is not sufficient for the CLT to hold!

(4) The old CLT for sums of i.i.d random variables $\zeta_k$ follows with $\xi_{n,k} := \zeta_k$.

(5) In a very precise sense: Condition (***) is sufficient and necessary for the CLT to hold (W. Feller).

*Proof.* W.l.o.g. we assume $(\forall n):\quad \sigma_n = 1$ and plan to prove:

$$(\forall u):\qquad \lim_{n\to\infty} \prod_{k=1}^{N_n} \varphi_{n,k}(u) = e^{-u^2/2}.$$

**Lemma.**

$$(\forall t \in \mathbb{R}):\qquad \left| e^{it} - \sum_{l=0}^{m} \frac{(it)^l}{l!} \right| \le \min\left\{ \frac{|t|^{m+1}}{(m+1)!}, \frac{2|t|^m}{m!} \right\}.$$

*Proof of the Lemma.* By induction on $m$:

$$\begin{aligned} e^{it} - \sum_{l=0}^{m} \frac{(it)^l}{l!} &= \frac{i^{m+1}}{m!} \int_0^t (t-s)^m e^{is} ds \\ &= \frac{i^m}{(m-1)!} \int_0^t (t-s)^{m-1} (e^{is} - 1) ds \end{aligned}$$

□

It follows that

$$
\begin{aligned}
\Big|\varphi_{n,k}(u) - 1 + \frac{u^2\sigma_{n,k}^2}{2}\Big| &= \Big|\mathbf{E}\Big( e^{iu\xi_{n,k}} - \sum_{l=0}^{2} \frac{(iu\xi_{n,k})^l}{l!}\Big)\Big| \\
&\leq \mathbf{E}\Big(\Big|e^{iu\xi_{n,k}} - \sum_{l=0}^{2} \frac{(iu\xi_{n,k})^l}{l!}\Big|\Big) \\
&\leq \mathbf{E}\Big( \min\{|u\xi_{n,k}|^3/6, |u\xi_{n,k}|^2\}\Big) \\
&\leq \frac{|u|^3}{6}\mathbf{E}\Big(|\xi_{n,k}|^3 \mathbb{1}_{|\xi_{n,k}|\leq\varepsilon}\Big) + \\
&\qquad + |u|^2\mathbf{E}\Big(|\xi_{n,k}|^2 \mathbb{1}_{|\xi_{n,k}|>\varepsilon}\Big) \\
&\leq \frac{\varepsilon|u|^3}{6}\sigma_{n,k}^2 + |u|^2\mathbf{E}\Big(|\xi_{n,k}|^2 \mathbb{1}_{|\xi_{n,k}|>\varepsilon}\Big).
\end{aligned}
$$

Hence, using (***),

$$\lim_{n\to\infty}\sum_{k=1}^{N_n}\left|\varphi_{n,k}(u)-1+\frac{u^2\sigma_{n,k}^2}{2}\right|=0. \tag{1}$$

This is the main point of the proof!

$$
\begin{aligned}
\Big|\sum_{k=1}^{N_n} \log\varphi_{n,k}(u) + \frac{u^2}{2}\Big| &= \Big|\sum_{k=1}^{N_n}\Big(\log\varphi_{n,k}(u) + \frac{u^2\sigma_{n,k}^2}{2}\Big)\Big| \\
&\le \sum_{k=1}^{N_n}\Big|\log\varphi_{n,k}(u) + \frac{u^2\sigma_{n,k}^2}{2}\Big| \\
&\le \sum_{k=1}^{N_n}\Big|\log\varphi_{n,k}(u) + (1-\varphi_{n,k}(u))\Big| + \\
&\qquad + \sum_{k=1}^{N_n}\Big|\varphi_{n,k}(u) - 1 + \frac{u^2\sigma_{n,k}^2}{2}\Big| \quad (2)
\end{aligned}
$$

We show that the last two sums go to zero, as $n \to \infty$.

From (1) it follows that

$$\limsup_{n\to\infty} \max_{1\le k\le N_n} \left|\varphi_{n,k}(u) - 1\right| \le$$

$$\le \limsup_{n\to\infty} \max_{1\le k\le N_n} \left|\varphi_{n,k}(u) - 1 + \frac{u^2\sigma_{n,k}^2}{2}\right| + \limsup_{n\to\infty} \max_{1\le k\le N_n} \frac{u^2\sigma_{n,k}^2}{2}$$

$$= 0. \tag{3}$$

This implies that for $n \ge n_0$ and $1 \le k \le N_n$ :

$$|\varphi_{n,k}(u) - 1\,| < 1/2,$$

and

$$\left|\log\varphi_{n,k}(u) + 1 - \varphi_{n,k}(u)\right| \le |1 - \varphi_{n,k}(u)\,|^2 \le$$

$$\le \left(\max_{1\le k'\le N_n} |1 - \varphi_{n,k'}(u)\,|\right)\left(\left|\varphi_{n,k}(u) - 1 + \frac{u^2\sigma_{n,k}^2}{2}\right| + \frac{u^2\sigma_{n,k}^2}{2}\right)$$

Hence

$$\sum_{k=1}^{N_n} \left| \log \varphi_{n,k}(u) + (1 - \varphi_{n,k}(u)) \right| \leq$$

$$\leq \left( \max_{1 \leq k' \leq N_n} |1 - \varphi_{n,k'}(u)| \right) \left( \sum_{k=1}^{N_n} \left| \varphi_{n,k}(u) - 1 + \frac{u^2 \sigma_{n,k}^2}{2} \right| + \frac{u^2}{2} \right)$$

Now, from (1), (2) and (3) it follows that

$$\lim_{n\to\infty} \left| \sum_{k=1}^{N_n} \log \varphi_{n,k}(u) + \frac{u^2}{2} \right| = 0.$$

□

**Application 1: CLT for the number of records.**
Let $\eta_k$, $k = 1, 2, \ldots$ be i.i.d., $\eta_k > 0$, with continuous distrib., and

$$\xi_1 = 1, \quad \xi_k := 1\!\!1_{\eta_k > \max_{1 \le j < k} \eta_j}, \quad k > 1, \quad S_n := \xi_1 + \cdots + \xi_n.$$

Then $\xi_k$, $k = 1, 2, \ldots$ are *independent* (HW!) with distribution

$$\mathbf{P}\big(\, \xi_k = 1 \,\big) = \frac{1}{k} = 1 - \mathbf{P}\big(\, \xi_k = 0 \,\big),$$

$$\mathbf{E}\big(\, \xi_k \,\big) = \frac{1}{k}, \qquad \mathbf{Var}\big(\, \xi_k \,\big) = \frac{k-1}{k^2},$$

$$\mathbf{E}\big(\, S_n \,\big) = \log n + \mathcal{O}(1), \qquad \mathbf{Var}\big(\, S_n \,\big) = \log n + \mathcal{O}(1).$$

**Theorem.**

$$\frac{S_n - \log n}{\sqrt{\log n}} \Rightarrow N(0, 1).$$

**Application 2: CLT in the "borderline" case.**
Let $\eta_k$, $k = 1, 2 \ldots$ i.i.d. with distribution density

$$\frac{d}{dx}\mathbf{P}\Big( \eta_j < x \Big) =: f(x) = |x|^{-3} 1\!\!1_{|x|>1}.$$

Then

$$(\forall \varepsilon > 0): \ \mathbf{E}\Big( |\eta_j|^{2-\varepsilon} \Big) < \infty, \qquad \mathbf{E}\Big( \eta_j \Big) = 0, \qquad \mathbf{E}\Big( |\eta_j|^2 \Big) = \infty.$$

**Theorem.**

$$\frac{\eta_1 + \cdots + \eta_n}{\sqrt{n \log n}} \Rightarrow N(0,1).$$

*Proof:* Define

$$\xi_{n,k} := \eta_k \mathbb{1}_{|\eta_k| < \sqrt{n} \log\log n}, \qquad k = 1, 2, \ldots, n$$

And apply Lindeberg's Theorem for the triangular array $\xi_{n,k}$, $k = 1, 2, \ldots n$, $n = 1, 2, \ldots$.

Mind that $(\forall n) : \xi_{n,k}$, $k = 1, 2, \ldots n$ are i.i.d.

$$\mathbf{E}\Big( \xi_{n,k} \Big) = 0,$$

$$\sigma_{n,k}^2 = \sigma_{n,1}^2 = \log n + 2 \log\log\log n,$$

$$\sigma_n^2 = n\sigma_{n,1}^2.$$

Lindeberg's condition (***):

$$\lim_{n\to\infty} \frac{1}{\sigma_{n,1}^2} \mathbf{E}\Big( |\xi_{n,1}|^2 \mathbb{1}_{|\xi_{n,1}|^2 > \varepsilon n \sigma_{n,1}^2} \Big) = 0$$

holds because

$$|\xi_{n,1}|^2 < n(\log\log n)^2 < \varepsilon n \sigma_{n,1}^2.$$

So:

$$\frac{\xi_{n,1} + \cdots + \xi_{n,n}}{\sqrt{n \log n}} \Rightarrow N(0,1). \tag{4}$$

What is the error made with the cutoff?

$$\mathbf{P}\Big( (\exists k \le n) : \xi_{n,k} \neq \eta_k \Big) \le n\mathbf{P}\Big( \xi_{n,1} \neq \eta_1 \Big) = \frac{1}{(\log\log n)^2} \to 0$$

Hence

$$\sum_{k=1}^{n} |\eta_k - \xi_{n,k}| \xrightarrow{\mathbf{P}} 0. \tag{5}$$

The theorem follows form (4) and (5). □

# VI.
# STABLE DISTRIBUTIONS AND STABLE LIMITS

**AFFINE EQUIVALENCE:**

**Definition:** *The probability distributions* $F_1, F_2 : \mathbb{R} \to [0,1]$ *are* *affine-equivalent* *iff*

$$\Big(\exists\ a \in (0,\infty), b \in \mathbb{R}\Big) : \quad \Big(\forall\ x \in \mathbb{R}\Big) : \quad F_2(x) = F_1(ax+b).$$

**Remarks:** (1) This is clearly an equivalence relation. A class of equivalence can be parametrized as

$$(0,\infty) \times (-\infty,\infty) \ni (a,b) \mapsto F_{a,b}(\cdot) := F_{1,0}(a \cdot + b).$$

(2) In terms of the random variables $X_1, X_2$ (of disrib. $F_1, F_2$):

$$\Big(\exists\ a \in (0,\infty), b \in \mathbb{R}\Big) : \quad X_2 \sim aX_1 + b.$$

(3) In terms of the characteristic functions $\varphi_1, \varphi_2$ (of the disributions $F_1, F_2$):

$$\Big(\exists\ a \in (0,\infty), b \in \mathbb{R}\Big) : \quad \Big(\forall\ u \in \mathbb{R}\Big) : \quad \varphi_2(u) = e^{ibu}\varphi_1(au).$$

**STABILITY:**

**Definition:** *An affine-equivalent class of distributions is stable iff it is closed under convolution.*
*A distribution is called stable if it belongs to a stable class: the distribution* $F$ *is stable iff*

$$\Big(\forall\, a_1, a_2 > 0\Big):\ \Big(\exists\, a_3 > 0, b_3 \in \mathbb{R}\Big):\ F(a_1 \cdot) * F(a_2 \cdot) = F(a_3 \cdot + b_3).$$

**Remarks:** (1) In terms of the random variables:

$$\Big(\forall\, a_1, a_2 > 0\Big):\ \Big(\exists\, a_3 > 0, b_3 \in \mathbb{R}\Big):\ a_1 X_1 + a_2 X_2 = a_3 X_3 + b_3,$$

where $X_1, X_2, X_3 \sim F$ and $X_1, X_2$ are *independent*.

(2) In terms of the characteristic function:

$$\Big(\forall\, a_1, a_2 > 0\Big):\ \Big(\exists\, a_3 > 0, b_3 \in \mathbb{R}\Big):\ \varphi(a_1 u)\varphi(a_2 u) = e^{i b_3 u}\varphi(a_3 u).$$

(3) By induction it follows that

$$\Big(\forall\ k \in \mathbb{N},\ a_1, \ldots, a_k > 0\Big) : \quad \Big(\exists\ a_{k+1} > 0,\ b_{k+1} \in \mathbb{R}\Big) :$$

$$a_1 X_1 + \cdots + a_k X_k = a_{k+1} X_{k+1} + b_{k+1},$$

where $X_1, \ldots, X_k, X_{k+1} \sim F$ and $X_1, \ldots, X_k$ are *independent*.

**EXAMPLES:**

**EX1:** (counterexample): Discrete distributions CAN'T BE stable. Actually: a stable distribution doesn't have point mass. (Obvious!)

**EX2:** The class of Gaussian (normal) distributions is stable:

$$\sigma > 0, m \in \mathbb{R}: \quad f_{\sigma,m}(x) := \frac{1}{\sqrt{2\pi}\sigma} \exp\{-(x-m)^2/(2\sigma^2)\},$$

$$\varphi_{\sigma,m}(u) = \exp\{imu - \frac{\sigma^2 u^2}{2}\}.$$

Indeed, for $\sigma_1, \sigma_2 > 0$ and $m_1, m_2 \in \mathbb{R}$

$$f_{\sigma_1,m_1} * f_{\sigma_2,m_2} = f_{\sigma_3,m_3}, \qquad \varphi_{\sigma_1,m_1}\varphi_{\sigma_2,m_2} = \varphi_{\sigma_3,m_3},$$

with

$$\sigma_3 = (\sigma_1^2 + \sigma_2^2)^{1/2}, \qquad m_3 = m_1 + m_2.$$

**EX3:** The class of Cauchy distributions is stable:

$$\tau > 0, m \in \mathbb{R}: \quad f_{\tau,m}(x) := \frac{1}{\pi\tau} \cdot \frac{1}{1 + (x-m)^2/\tau^2},$$

$$\varphi_{\tau,m}(u) = \exp\{imu - \tau|u|\}.$$

Indeed, for $\tau_1, \tau_2 > 0$ and $m_1, m_2 \in \mathbb{R}$

$$f_{\tau_1,m_1} * f_{\tau_2,m_2} = f_{\tau_3,m_3}, \qquad \varphi_{\tau_1,m_1}\varphi_{\tau_2,m_2} = \varphi_{\tau_3,m_3},$$

with

$$\tau_3 = \tau_1 + \tau_2, \qquad m_3 = m_1 + m_2.$$

**EX4:** Recall the distribution of first hitting times of Brownian motion: $B_t$ standard $1d$ Brownian motion, $B_0 = 0$.

$$T_r := \inf\{t : B_t = r\} \overset{(1)}{\sim} r^2 \inf\{t : B_t = 1\}$$

$$f_{\alpha,0}(s) := \frac{\partial}{\partial s}\mathbf{P}\Big(T_{\sqrt{\alpha}} < s\Big) = \frac{\partial}{\partial s}\mathbf{P}\Big(T_1 < s/\alpha\Big).$$

(1): By scaling of Brownian motion.

Then,

$$\alpha > 0, m = 0: \quad f_{\alpha,0}(s) \overset{(2)}{=} \frac{1}{\sqrt{2\pi\alpha}}(s/\alpha)^{-3/2}e^{-\alpha/(2s)}\mathbb{1}_{s>0},$$

$$\varphi_{\alpha,0}(u) \overset{(3)}{=} \exp\{-(1+i)\alpha^{1/2}|u|^{1/2}\}.$$

(2): See earlier work. (Max. and hitting times of RW and BM.)

(3): Computation. Will be done later. Try it as HW.

Indeed, for $\alpha_1, \alpha_2 > 0$

$$f_{\alpha_1,0} * f_{\alpha_2,0} \overset{(4)}{=} f_{\alpha_3,0}, \qquad \varphi_{\alpha_1,0}\varphi_{\alpha_2,0} = \varphi_{\alpha_3,0},$$

with

$$\alpha_3 = (\alpha_1^{1/2} + \alpha_2^{1/2})^2.$$

(4): By independent + stationary increments and scaling of Brownian motion:

$$(r_1 + r_2)^2 T_1 \ \sim\ T_{r_1+r_2} \ \sim\ T_{r_1} + T'_{r_2} \ \sim\ r_1^2 T_1 + r_2^2 T'_1$$

where $T_{r_1}$ and $T'_{r_2}$, respectively, $T_1$ and $T'_1$ are *independent*.

HW: Let $(X_t, Y_t)$ be standard $2d$ Brownian motion, starting form $(X_0, Y_0) = (0,0)$, and $T_1 := \inf\{t : X_t = 1\}$. Compute the distribution of $Y_{T_1}$.

**Proposition.** *The distribution* $F$ *is stable if and only if for any* $k \in \mathbb{N}$ *there exist* $\alpha_k > 0, \beta_k \in \mathbb{R}$ *such that*

$$X_1 + \cdots + X_k = \alpha_k X + \beta_k,$$

*where* $X_1, \ldots, X_k, X \sim F$ *and* $X_1, \ldots, X_k$ *are independent.*

*Proof of the Proposition: Later.* □

Limit laws of centred and normed sums of i.i.d. random variables are always stable:

**Theorem.** *Let* $X_1, X_2, \ldots$ *be i.i.d. random variables and* $S_n := X_1 + \cdots + X_n$. *If there exist (deterministic) sequences* $a_n > 0$ *and* $b_n \in \mathbb{R}$ *such that*

$$Z_n := \frac{S_n - b_n}{a_n} \Rightarrow Y,$$

*as* $n \to \infty$, *then the distribution of* $Y$ *is stable.*

*Proof.* Fix $k \in \mathbb{N}$ and denote,

$$\sum_{i=n(j-1)+1}^{nj} X_i =: S_n^{(j)} \sim S_n. \qquad j = 1, \ldots, k.$$

Then

$$S_{kn} = S_n^{(1)} + \cdots + S_n^{(k)}$$

and

$$\frac{a_{kn}}{a_n} Z_{kn} - \frac{kb_n - b_{kn}}{a_n} = Z_n^{(1)} + \cdots + Z_n^{(k)},$$

where $Z_n^{(1)}, \cdots, Z_n^{(k)} \sim Z_n$ are i.i.d. Thus:

$$Z_{kn} \Rightarrow Y, \qquad Z_n^{(1)} + \cdots + Z_n^{(k)} \Rightarrow Y^{(1)} + \cdots + Y^{(k)},$$

as $n \to \infty$, where $Y^{(1)}, \ldots, Y^{(k)} \sim Y$ are i.i.d.

**Lemma.** *Let* $W_n$ *be a sequence of random variables,* $\alpha_n > 0$, $\beta_n \in \mathbb{R}$ *(deterministic) sequences, and* $W_n' := \alpha_n W_n + \beta_n$. *If both* $W_n \Rightarrow W$ *and* $W_n' \Rightarrow W'$, *as* $n \to \infty$ *where* $W$ *and* $W'$ *are both nondegenerate random variables, then the limits* $\lim_{n\to\infty} \alpha_n =: \alpha > 0$ *and* $\lim_{n\to\infty} \beta_n =: \beta \in \mathbb{R}$ *exist.*

*Proof of the Lemma.* Easy: write the characteristic functions. □

By the Lemma the limits

$$\lim_{n\to\infty} \frac{a_{kn}}{a_n} =: \alpha_k > 0, \qquad \lim_{n\to\infty} \frac{kb_n - b_{kn}}{a_n} =: \beta_k \in \mathbb{R}$$

exist, and thus

$$Y^{(1)} + \cdots + Y^{(k)} \sim \alpha_k Y - \beta_k.$$

The Theorem follows from the Proposition. □

**SYMMETRIC STABLE LAWS** (easier then the general case)

**Theorem.** *(i) Let* $c > 0$ *and* $\alpha \in (0, 2]$. *The function* $\varphi : \mathbb{R} \to \mathbb{R}$

$$\varphi(u) := \exp\{-c|u|^{\alpha}\} \tag{6}$$

*is characteristic function of a symmetric stable distribution.*
*(ii) The characteristic function of a symmetric stable distribution is of the form* (6), *with some* $c > 0$ *and* $\alpha \in (0, 2]$.

**Remarks:** (1) The parameter $c > 0$ can be changed by scaling. The parameter $\alpha \in (0, 2]$ is *essential*. It is called the index of the stable law.

(2) $u \mapsto \varphi(u)$ of (6) obviously satisfies the stability condition. It is to be checked that

– It is indeed a characteristic function.

– There are no other chf-s of symmetric stable laws.

(3) Examples: $\alpha = 2$: Gaussian; $\alpha = 1$: Cauchy.
No explicit formula for the distribution function/density in other cases.

(4) In the symmetric stable case:

$$a_1 X_1 + a_2 X_2 \ \sim\ \left(a_1^\alpha + a_2^\alpha\right)^{1/\alpha} X.$$

*Proof of (i), for $\alpha \in (0, 1]$:*
**Theorem** (György Pólya's construction). *Let $\varphi : \mathbb{R} \to [0, 1]$ satisfy:*

- $\lim_{u \to 0} \varphi(u) = 1$, • $\varphi(-u) = \varphi(u)$, • $[0, \infty) \ni u \mapsto \varphi(u)$ *convex.*

*Then $\varphi$ is a characteristic function.*

If $\alpha \in (0, 1]$, then $\varphi(u)$ of (6) is of this form. □

*Proof of Pólya's theorem.*

$$\psi_1(u) := \left(1 - |u|\right)_+ = \int_{-\infty}^{\infty} e^{iux} \, \frac{1 - \cos x}{\pi x^2} dx,$$

$$\psi_a(u) := \psi_1(au) = \left(1 - a|u|\right)_+, \qquad a > 0,$$

are characteristic functions. The functions of the theorem are *pointwise limits* of functions of the form

$$u \mapsto \sum_{k=1}^{K} p_k \psi_{a_k}(u),$$

with

$$a_1, \ldots, a_K > 0; \qquad p_1, \ldots, p_K \in [0, 1], \quad p_1 + \cdots + p_K = 1,$$

which are themselves characteristic functions. □

*Proof of (i), for $\alpha \in (0,2)$:*
Let $X_1, X_2, \ldots$ be i.i.d. with symmetric distribution density $f$:

$$f(x) := \frac{\alpha}{2|x|^{\alpha+1}} 1\!\!1_{|x|>1},$$

and characteristic function $\psi$. Then

$$1 - \psi(u) = \alpha \int_1^\infty \frac{1-\cos(ux)}{x^{\alpha+1}} dx = \alpha |u|^\alpha \int_{|u|}^\infty \frac{1-\cos y}{y^{\alpha+1}} dy$$

Since $0 < \alpha < 2$ (!):

$$\int_0^\infty \frac{1-\cos y}{y^{\alpha+1}} dy =: \frac{c}{\alpha} < \infty,$$

$$\int_0^{|u|} \frac{1-\cos y}{y^{\alpha+1}} dy = \mathcal{O}(|u|^{2-\alpha}).$$

Thus,

$$\psi(u) = 1 - c|u|^\alpha + \mathcal{O}(|u|^2),$$

and hence, for any $u \in \mathbb{R}$ fixed

$$\mathbf{E}\Big( \exp\{iu\frac{S_n}{n^{1/\alpha}}\}\Big) = \psi(un^{-1/\alpha})^n$$

$$= \Big(1 - \frac{c|u|^\alpha}{n} + \mathcal{O}(n^{-2/\alpha})\Big)^n \to e^{-c|u|^\alpha}.$$

We have proved that $u \mapsto e^{-c|u|^\alpha}$ is the characteristic function of a symmetric stable distribution $F$ and the limit theorem

$$\lim_{n\to\infty} \mathbf{P}\Big(\frac{S_n}{n^{1/\alpha}} < x\Big) = F(x).$$

□

*Proof of (ii).* We prove that if $\varphi$ is characteristic function of a symmetric stable law then it is of the form (6).

Let $F$ be a symmetric stable law and $\varphi$ its characteristic function.

**Lemma** (Some basic facts about $\varphi$). *(i)*

$$\Big(\forall u \in \mathbb{R}\Big): \quad \varphi(u) = \overline{\varphi(u)} = \varphi(-u).$$

*(ii)*

$$\Big(\forall u \in \mathbb{R}\Big): \quad \varphi(u) > 0.$$

*(iii) If* $b > a > 0$ *then*

$$\Big(\exists u \in \mathbb{R}\Big): \quad \varphi(bu) \neq \varphi(au).$$

*Proof of the Lemma.*

(i) follows from symmetry of the distribution $F$.

(ii) Due to symmetry and stability,

$$\Big(\exists\ c \in (0,1) \cup (1,\infty)\Big) :\ \Big(\forall\ u \in \mathbb{R}\Big) :\ \varphi(u)^2 = \varphi(cu)$$

(If $c = 1$ then $(\forall\ u \in \mathbb{R}) :\ \varphi(u)^2 = \varphi(u)$, and, by continuity at $u = 0$, $\varphi(u) \equiv 1$. This case is excluded.)

$$\Big\{\varphi(u_0) = 0\Big\} \ \Rightarrow\ \Big\{\Big(\forall\ k \in \mathbb{Z}\Big) :\ \varphi(c^k u_0) = 0\Big\}.$$

This is impossible, due to continuity at $u = 0$.

(iii) This holds for any characteristic function.
Let $c := a/b < 1$. By continuity at $u = 0$

$$\Big\{\Big(\forall\ u \in \mathbb{R}\Big) :\ \varphi(u) = \varphi(cu)\Big\} \ \Rightarrow\ \Big\{\varphi(u) \equiv 1\Big\}.$$

But this case is excluded. □

By symmetric stability there exists

$$\gamma : \mathbb{N} \to \mathbb{R}_+, \qquad \Big(\forall\ u \in \mathbb{R}\Big): \ \ \varphi(u)^n = \varphi(\gamma(n)u).$$

We get

$$\Big(\forall\ u \in \mathbb{R}\Big): \quad \varphi(\gamma(nm)u) = \varphi(u)^{nm} = \varphi(\gamma(n)\gamma(m)u)$$

Hence, by (iii) of the Lemma

$$\gamma(nm) = \gamma(n)\gamma(m).$$

Extend

$$\gamma : \mathbb{Q} \to \mathbb{R}_+, \qquad \gamma(n/m) := \gamma(n)/\gamma(m).$$

Then

$$\Big(\forall\ u \in \mathbb{R}\Big): \quad \varphi(u)^{n/m} = \cdots = \varphi(\gamma(n/m)u)$$

Let $r \in \mathbb{R}_+$ and $r_n \in \mathbb{Q}$, $\lim_{n\to\infty} r_n = r$.

Then

$$\varphi(\gamma(r_n)u) = \varphi(u)^{r_n} \to \varphi(u)^r.$$

$\gamma(r_{n'}) \to 0$ or $\gamma(r_{n'}) \to \infty$ implies $\varphi(u) \equiv 1$ – impossible. Similarly, if $\gamma(r_{n'}) \to g' \in \mathbb{R}$ and $\gamma(r_{n''}) \to g'' \in \mathbb{R}$ then again by (iii) of the Lemma $g' = g''$. So, we extend $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$, such that

$$\Big(\forall\ u \in \mathbb{R}\Big):\ \ \varphi(u)^r = \varphi(\gamma(r)u) \tag{7}$$

$$\gamma(rs) = \gamma(r)\gamma(s). \tag{8}$$

$$r \mapsto \gamma(r) \qquad \text{is continuous.} \tag{9}$$

**Lemma** (“Cauchy’s problem”). *Let* $\gamma : (0, \infty) \to (0, \infty)$ *satisfy* (8) *and* (9). *Then* $\gamma(r) = r^\beta$ *for some* $\beta \in \mathbb{R}$.

From (7) it follows that $\varphi(u) = \exp\{-c|u|^\alpha\}$ with $\alpha = 1/\beta$.
$c \leq 0$ or $\alpha \notin (0, 2]$ are a priori excluded. □

**Remarks:**

(1) The symmetric stable distributions are absolutely continuous with $C^\infty$ density functions.

(2) "Heavy tail" : For $\alpha \in (0,2)$:

$$F'(x) =: f(x) \sim C(\alpha)|x|^{-\alpha-1}, \qquad \text{as} \qquad |x| \to \infty$$

(3) In particular

$$(\forall \varepsilon > 0): \quad \mathbf{E}\Big(|X|^{\alpha-\varepsilon}\Big) < \infty,$$
$$\mathbf{E}\Big(|X|^{\alpha}\Big) = \infty.$$

**Theorem.** *Let* $X_1, X_2, \ldots$ *be i.i.d. random variables. Denote their (common) distribution function by* $F$ *and* $S_n := X_1 + \cdots + X_n$*. Assume that the distribution* $F$ *is symmetric*

$$F(-x) = 1 - F(x+0),$$

*and the tail of the distribution has regular power-law asymptotics*

$$\lim_{x \to \infty} x^{\alpha}\Big(1 - F(x)\Big) = b,$$

*with* $\alpha \in (0, 2)$ *and* $b \in (0, \infty)$*. Then*

$$\lim_{n \to \infty} \mathbf{E}\Big( \exp\{iuS_n/n^{1/\alpha}\} \Big) = e^{-c|u|^{\alpha}},$$

*with*

$$c = 2b\alpha \int_0^{\infty} \frac{1 - \cos y}{y^{\alpha+1}} dy.$$

**Remark:** This Theorem extends the earlier construction. A more general Theorem will be stated later.

*Proof.* We prove for $|u| \ll 1$

$$\psi(u) := \mathbf{E}\Big( \text{exp}\{iuX_j\} \Big) = 1 - c|u|^{\alpha} + o(|u|^{\alpha}), \qquad (***)$$

and hence

$$\begin{aligned}\mathbf{E}\Big( \text{exp}\{iuS_n/n^{1/\alpha}\} \Big) &= \Big(\psi(u/n^{1/\alpha})\Big)^n \\ &= \Big(1 - c|u|^{\alpha}/n + o(1/n)\Big)^n \to e^{-c|u|^{\alpha}}.\end{aligned}$$

Proof of (***) follows:
Fix $\varepsilon > 0$, at the end of the proof we let $\varepsilon \to 0$.

$$1 - \psi(u) = 2\int_0^{\infty} \Big(1 - \cos(ux)\Big) dF(x)$$

$$= 2\int_0^{1/(\varepsilon u)} \Big(1 - \cos(ux)\Big) dF(x) + 2\int_{1/(\varepsilon u)}^{\infty} \Big(1 - \cos(ux)\Big) dF(x).$$

Further,

$$2\int_0^{1/(\varepsilon u)} \Big(1 - \cos ux\Big) dF(x) = 2\int_0^{1/(\varepsilon u)} \Big(1 - \cos ux\Big) d(F(x) - 1) =$$

$$= 2\Big(1 - \cos(1/\varepsilon)\Big)\Big(F(1/(\varepsilon u)) - 1)\Big) + 2u\int_0^{1/(\varepsilon u)} \Big(1 - F(x)\Big) \sin(ux) dx.$$

Altogether

$$1 - \psi(u) = A(u,\varepsilon) + B(u,\varepsilon) + C(u,\varepsilon),$$

where

$$A(u,\varepsilon) := 2\int_{1/(\varepsilon u)}^{\infty} \Big(1 - \cos(ux)\Big) dF(x),$$

$$B(u,\varepsilon) := 2\Big(1 - \cos(1/\varepsilon)\Big)\Big(F(1/(\varepsilon u)) - 1)\Big),$$

$$C(u,\varepsilon) := 2u\int_{0}^{1/(\varepsilon u)} \Big(1 - F(x)\Big) \sin(ux) dx.$$

We keep $\varepsilon > 0$ fixed. Then clearly,

$$\max\{|A(u,\varepsilon)|, |B(u,\varepsilon)|\} \le 4\Big(1 - F(1/(\varepsilon u))\Big) = 4b\varepsilon^{\alpha}|u|^{\alpha} + o(|u|^{\alpha}).$$

$$
\begin{aligned}
C(u,\varepsilon) &\overset{(1)}{=} 2|u|^\alpha \int_0^{1/\varepsilon} (y/u)^\alpha \Big(1 - F(y/u)\Big) \frac{\sin y}{y^\alpha} dy \\
&\overset{(2)}{=} 2b|u|^\alpha \int_0^{1/\varepsilon} \frac{\sin y}{y^\alpha} dy + o(|u|^\alpha) \\
&\overset{(3)}{=} 2b\alpha|u|^\alpha \int_0^{1/\varepsilon} \frac{1-\cos y}{y^{\alpha+1}} dy + 2b|u|^\alpha \varepsilon^\alpha \Big(1 - \cos(1/\varepsilon)\Big) + o(|u|^\alpha) \\
&\overset{(4)}{=} 2b\alpha|u|^\alpha \int_0^{\infty} \frac{1-\cos y}{y^{\alpha+1}} dy - 2b\alpha|u|^\alpha \int_{1/\varepsilon}^{\infty} \frac{1-\cos y}{y^{\alpha+1}} dy + \\
&\qquad + 2b|u|^\alpha \varepsilon^\alpha \Big(1 - \cos(1/\varepsilon)\Big) + o(|u|^\alpha)
\end{aligned}
$$

(1): Change of variable $y := ux$ (2): Dominated convergence.
(3): Integration by parts. (4): Absolute integrability.

Altogether, with *any* $\varepsilon > 0$ fixed:

$$\left|1 - \psi(u) - c|u|^\alpha\right| \le o(|u|^\alpha) + 16b\varepsilon^\alpha|u|^\alpha.$$

Hence (***). □

## EXAMPLES, APPLICATIONS

**EX 1: Sums of reciprocals of absolutely continuous i.i.d. r.v.-s.**

Let $X_1, X_2, \ldots$ be i.i.d. random variables with absolutely continuous distribution. Denote their density function $f$ and assume that $f$ is continuous at $x = 0$ and $f(0) \in (0, \infty)$. Then

$$\frac{1}{n} \sum_{k=1}^{n} \frac{1}{X_k} \Rightarrow CAU(0, \tau),$$

with some $\tau \in (0, \infty)$.

**EX 2: Holtzmark's first (one dimensional) problem.**

For $n \in \mathbb{N}$, let $X_{n,1}, \ldots, X_{n,n}$ be i.i.d. $UNI[-n/2, n/2]$. These are "positions of stars or charges". A star/charge located at $x \in \mathbb{R}$ generates at the origin the force

$$F(x) = \mathsf{sgn}(x)|x|^{-p}.$$

So, the resulting *total force* generated by the system of $n$ randomly positioned stars at the origin is

$$R_n = \sum_{k=1}^{n} \mathsf{sgn}(X_{n,k})|X_{n,k}|^{-p}.$$

Question: Does $R_n$ have a limiting distribution, as $n \to \infty$?

**Theorem.** *If* $1/2 < p < \infty$ *then*

$$\lim_{n\to\infty} \mathbf{E}\Big( \exp\{iuR_n\} \Big) = e^{-c|u|^{1/p}}$$

*with*

$$c = HW!.$$

*Proof.*
Let $Y_1, Y_2, \ldots$ be i.i.d. $UNI[-1/2, 1/2]$-distributed. Then

$$\{X_{n,1}, \ldots, X_{n,n}\} \sim \{nY_1, \ldots, nY_n\}$$

$$R_n \sim n^{-p} \sum_{k=1}^{n} \text{sgn}(Y_k)|Y_k|^{-p}.$$

Note that $\xi_k := \text{sgn}(Y_k)|Y_k|^{-p}$, $k = 1, 2, 3, \ldots$ are i.i.d., symmetric and

$$\mathbf{P}\Big(\text{sgn}(Y_k)|Y_k|^{-p} > x\Big) = \mathbf{P}\Big(0 < Y_k < x^{-1/p}\Big) = x^{-1/p}$$

and the limit theorem is applied. □

**EX 3: Holtzmark's second (multi-dimensional) problem.** Identical stars/charges are located in $\mathbb{R}^d$ according to a homogeneous Poisson point process (PPP) of density $\rho$. Denote their locations $\vec{X}_1^{(\rho)}, \vec{X}_2^{(\rho)}, \ldots$ in some (arbitrary) ordering. A star/charge located at $\vec{x} \in \mathbb{R}^d \setminus \vec{0}$ generates at the origin the force

$$\vec{F}(\vec{x}) = |\vec{x}|^{-p-1}\vec{x} = \vec{\mathrm{sgn}}(\vec{x})|\vec{x}|^{-p}.$$

Denote by $\vec{R}^{(\rho)}$ the resulting total force generated at the origin by all stars/charges. Formally:

$$\vec{R}^{(\rho)} = \sum_i \vec{F}(\vec{X}_i^{(\rho)}).$$

Note that convergence problems may arise.

Question: Assuming that $\vec{R}^{(\rho)}$ makes some sense, can we say something about its distribution?

1. **Scaling:**

$$\left(\vec{X}_1^{(\rho)}, \vec{X}_2^{(\rho)}, \ldots\right) \sim \left(\rho^{-1/d}\vec{X}_1^{(1)}, \rho^{-1/d}\vec{X}_2^{(1)}, \ldots\right)$$

$$(\forall a > 0): \qquad \vec{F}(a\vec{x}) = a^{-p}\vec{F}(\vec{x})$$

It follows that

$$\vec{R}^{(\rho)} \sim \rho^{p/d}\vec{R}^{(1)} \tag{10}$$

2. **"Independent increments":**
If $PPP^{(\rho_1)}$ and $PPP^{(\rho_2)}$ are two *independent* Poisson point processes of density $\rho_1$, respectively, $\rho_2$ then

$$PPP^{(\rho_1)} \cup PPP^{(\rho_2)} \sim PPP^{(\rho_1+\rho_2)}. \tag{11}$$

From (10) and (11) it follows that:

$$\rho_1^{p/d}\vec{R}' + \rho_2^{p/d}\vec{R}'' = (\rho_1 + \rho_2)^{p/d}\vec{R}'''$$

where $\vec{R}' \sim \vec{R}'' \sim \vec{R}''' \sim \vec{R}^{(1)}$, $\vec{R}'$ and $\vec{R}''$ are independent, and $\rho_1, \rho_2 > 0$.

**If $\vec{R}^{(\rho)}$ does make sense then it has symmetric stable distribution of index**

$$\alpha = \frac{d}{p}, \qquad \frac{d}{2} < p < \infty.$$

**Remarks.**

(1) The summation should be done as

$$\vec{R} = \lim_{n \to \infty} \sum_i \vec{F}(\vec{X}_i) 1\!\!1_{\vec{X}_i \in \Lambda_n},$$

where $\Lambda_n$ is a sequence of increasing, *symmetric* domains, $\cup_n \Lambda_n = \mathbb{R}^d$.

(2) If $p > d/2$ then the limit exists a.s., if $p \leq d/2$ then far-away charges/stars have divergent effect.

(3) Case of Coulomb or gravitational forces: In

$$d \geq 3: \qquad p = d - 1 > d/2, \qquad \mathrm{O}K!$$

**Towards more general limit theorems:**

**Definition.** *The function* $L : (0, \infty) \to (0, \infty)$ *is* *slowly varying* *(at infinity) iff*

$$(\forall a > 0) : \qquad \lim_{x \to \infty} \frac{L(ax)}{L(x)} = 1.$$

**Examples, remarks, HWs:**

(1) If $\lim_{x \to \infty} L(x) = b \in (0, \infty)$ then obviously $L$ is s.v.

(2) For any $\beta \in \mathbb{R}$, $L(x) := (\log x)^\beta$ is s.v.

(3) Show that for $\beta < 1$ and $c \in \mathbb{R}$, $L(x) := \exp\{c(\log x)^\beta\}$ is s.v.

(4) Construct a s.v. function $L$ for which

$$\liminf_{x \to \infty} L(x) = 0, \qquad \limsup_{x \to \infty} L(x) = \infty.$$

**Theorem.** *Let* $X_1, X_2, \ldots$ *be i.i.d. with symmetric distribution* $F$ *for which*

$$1 - F(x) = x^{-\alpha} L(x), \qquad \text{as} \quad x \to \infty,$$

*where* $\alpha \in (0, 2]$ *and* $L(x)$ *is slowly varying at infinity. Let*

$$a_n := \inf\{x : 1 - F(x) \leq 1/n\}.$$

*Then*

$$\lim_{n\to\infty} \mathbf{E}\Big( \exp\{iuS_n/a_n\} \Big) = \exp\{-c|u|^\alpha\}$$

*with some* $c \in (0, \infty)$.

**Remarks:**

(1) This extends (quite far) the previous limit theorems.

(2) The proof is more technical. We omit it.

## WITHOUT SYMMETRY:

Stable distributions are parametrized by:

- the index $\alpha \in (0, 2]$;
- the skewness $\kappa \in [-1, 1]$;
- the scale $c \in (0, \infty)$
- the shift $b \in \mathbb{R}$.

**Remarks:**

(1) The scale and shift change with affine transformations. We will choose them later $c = 1$, $b = 0$.

(2) The index and skewness are *relevant*.

(3) Notation: $STAB(\alpha, \kappa, c, b)$

**Theorem.** *The characteristic functions of stable distributions are*

$$\alpha \neq 1: \qquad \varphi(u) = \exp\left\{ibu - c|u|^{\alpha}\Big(1 - i\,\mathrm{sgn}(u)\,\kappa\,\tan\frac{\alpha\pi}{2}\Big)\right\}$$

$$\alpha = 1: \qquad \varphi(u) = \exp\left\{ibu - c|u|\Big(1 + i\,\mathrm{sgn}(u)\,\kappa\,\frac{2\log|u|}{\pi}\Big)\right\}.$$

**Remarks:**

(1) *Symmetric* stable laws: $\kappa = 0$.

(2) No skewness for $\alpha = 2$.

(3) All stable laws are absolutely continuous with $C^{\infty}$ density. Follows from fast decay of the chf. as $|u| \to \infty$.

(4) No explicit formula for the distribution/density function, except for the (already known) cases:
$\alpha = 2$ (Gauss); $\alpha = 1, \kappa = 0$ (Cauchy); $\alpha = 1/2, \kappa = \pm 1$ (Lévy).

(5) Let $X, Y \sim STAB(\alpha, \kappa = 1, c = 1, b = 0)$ be i.i.d. and $p,\ q \geq 0$ such that $p^{-\alpha} + q^{-\alpha} = 1$. Then

$$pX - qY \sim STAB(\alpha, \kappa = p^{-\alpha} - q^{-\alpha}, c = 1, b = 0).$$

(6) "Heavy tail": From the type of singularity of $\varphi(u)$ at $u = 0$ it follows that

$$(\forall \varepsilon > 0): \quad \mathbf{E}\Big( |X|^{\alpha - \varepsilon} \Big) < \infty,$$

$$(\forall \varepsilon \geq 0): \quad \mathbf{E}\Big( |X|^{\alpha + \varepsilon} \Big) = \infty.$$

More precisely:

$$\mathbf{P}\Big( |X| > x \Big) \sim C x^{-\alpha}$$

(7) **"Lower tail" in the totally skew ($\kappa = 1$) case:**

Fix: $b = 0$ (✓), $c = 1$ (✓), $\kappa = 1$ (!):

Then the chf

$$\alpha \neq 1: \qquad \varphi(u) = \exp\Big\{ -|u|^\alpha \Big(1 - i\,\mathrm{sgn}(u)\,\tan\frac{\alpha\pi}{2}\Big)\Big\}$$

$$\alpha = 1: \qquad \varphi(u) = \exp\Big\{ -|u| \Big(1 + i\,\mathrm{sgn}(u)\,\frac{2\log|u|}{\pi}\Big)\Big\}.$$

can be continued analytically into the complex upper half-plane $\mathbb{C}_+ := \{z \in \mathbb{C} : \mathfrak{Re}(z) > 0\}$:, as $\varphi_+ : \mathbb{C}_+ \to \mathbb{C}$,

$$\alpha \neq 1: \qquad \varphi_+(z) = \exp\Big\{ -\cos(\alpha\pi/2)^{-1}\,(-i\,z)^\alpha \Big\}$$

$$\alpha = 1: \qquad \varphi_+(z) = \exp\Big\{ \Big(i - \frac{2}{\pi}\log(z)\Big)\,(i\,z) \Big\}$$

No analytic continuation (matching both halflines $u > 0$ and $u < 0$) into the lower half plane!

**Theorem.** *Let* $X \sim STAB(\alpha, \kappa = +1, c = 1, b = 0)$. *The "moment generating function"* $\widetilde{\varphi} : [0, \infty) \to \mathbb{R}_+$

$$\widetilde{\varphi}(u) := \mathbf{E}\Big( \exp\{-uX\} \Big) = \varphi_+(iu)$$

*is*

$$\alpha \neq 1 : \qquad \widetilde{\varphi}(u) = \exp\Big\{ -\cos(\alpha\pi/2)^{-1}\, u^{\alpha} \Big\}$$

$$\alpha = 1 : \qquad \widetilde{\varphi}(u) = \exp\Big\{ \frac{2}{\pi} u \log(u) \Big\}.$$

*Proof.* Just done. □

**Corollary.** *(i) For* $\alpha \in (0,1)$*:*

$$\mathbf{P}\Big( X > 0 \Big) = 1,$$

$$\frac{d}{dx}\mathbf{P}\Big( 0 < X < x \Big) \sim c(\alpha) e^{-1/x} x^{-(\alpha+1)}, \quad \text{as} \quad x \to 0.$$

*(ii) For* $\alpha = 1$*:*

$$\mathbf{P}\Big( X < -x \Big) < \exp\{-ce^{x}\}$$

*(iii) For* $\alpha \in (1,2)$*:*

$$\mathbf{P}\Big( X < -x \Big) < \exp\{-cx^{\alpha/(\alpha-1)}\}$$

*Proof.* Tauberian arguments. . . . □

**Theorem** (Limit theorem in the non-symmetric case.). *Let* $X_1, X_2, \dots$ *be i.i.d. random variables. Assume*

$$(1) \qquad \mathbf{P}\Big(|X_j| > x\Big) = x^{-\alpha} L(x) \qquad \textit{with} \qquad \alpha \in (0,2),$$

$$(2) \qquad \lim_{x \to +\infty} \frac{\mathbf{P}\Big(X_j > x\Big)}{\mathbf{P}\Big(X_j < -x\Big)} =: \frac{1+\kappa}{1-\kappa} \in [0, \infty] \quad \textit{exists.}$$

*Define*

$$a_n := \inf\{x : \mathbf{P}\Big(|X_j| > x\Big) < n^{-1}\},$$

$$b_n := n\mathbf{E}\Big(X_j \mathbb{1}_{|X_j| \le a_n}\Big).$$

*Then*

$$\frac{S_n - b_n}{a_n} \Rightarrow STAB(\alpha, \kappa, c, b),$$

*with some* $c \in (0, \infty)$, $b \in \mathbb{R}$.

**Remark:** Note that

$$a_n = n^{1/\alpha}\widetilde{L}(n),$$

with $\widetilde{L}$ slowly varying at infinity.

# VII.
# INFINITELY DIVISIBLE DISTRIBUTIONS

## INFINITE DIVISIBILITY:

**Definition:** *The probability distribution* $F$ *is infinitely divisible iff for any* $n \in \mathbb{N}$ *there exists a probability distribution* $F_n$ *so that*

$$F = \left(F_n\right)^{*n}.$$

**Remarks:**

(1) In terms of the random variables: $X$ is infinitely divisible iff for any $n \in \mathbb{N}$ there are $X_{n,1}, X_{n,2}, \dots, X_{n,n}$ i.i.d. so that

$$X \sim X_{n,1} + X_{n,2} + \cdots + X_{n,n}.$$

(2) In terms of the characteristic functions: $\varphi(u)$ is an infinitely divisible chf. iff for any $n \in \mathbb{N}$ there exists a chf. $\varphi_n(u)$ so that

$$\varphi(u) = \left(\varphi_n(u)\right)^n.$$

**EXAMPLES:**

**EX1:** The normal distribution:

$$N(0, \sigma^2 = t) = N(0, \sigma^2 = t/n)^{*n}.$$

$$f_t(y) = \frac{1}{\sqrt{2\pi t}} e^{-y^2/(2t)}, \qquad \varphi_t(u) = \exp\{-tu^2/2\}$$

**EX2:** The Cauchy distributions:

$$CAU(0, \tau = t) = N(0, \tau = t/n)^{*n}.$$

$$f_t(y) = \frac{1}{\pi} \frac{t}{t^2 + y^2}, \qquad \varphi_t(u) = \exp\{-t|u|\}$$

**EX3:** Stable distributions in general: If $X \sim STAB(\alpha, \kappa, c = t, b = 0)$, and $X_{n,1}, \ldots, X_{n,n} \sim STAB(\alpha, \kappa, c = t/n, b = 0)$ are i.i.d. Then

$$X \sim n^{-\alpha}\Big(X_{n,1} + \cdots + X_{n,n}\Big) = \frac{X_{n,1}}{n^\alpha} + \cdots + \frac{X_{n,n}}{n^\alpha}.$$

The density and characteristic functions (for $\alpha \neq 1$):

$$f_t(y) = ???, \qquad \varphi_t(u) = \exp\Big\{ - t|u|^\alpha \Big(1 - i\,\mathrm{sgn}(u)\,\kappa\,\tan\frac{\alpha\pi}{2}\Big)\Big\}.$$

**EX4:** Poisson (Not stable!):

$$POI(\varrho t) = POI(\varrho t/n)^{*n}.$$

$$p_t(k) = e^{-t\rho}\frac{(t\rho)^k}{k!}, \qquad \varphi_t(u) = \exp\Big\{t\rho(e^{iu} - 1)\Big\}$$

**EX5:** The gamma distributions (Not stable!):

$$GAM(t) = GAM(t/n)^{*n}.$$

$$f_t(y) = \Gamma(t)^{-1} e^{-y} y^{t-1}, \qquad \varphi_t(u) = \exp\Big\{ -t \log(1 - iu) \Big\}.$$

**EX6:** The negative binomial distributions (Not stable!):

$$NB(p,t) = NB(p,t/n)^{*n}.$$

$$p_t(k) = (-1)^k \binom{-t}{k} (1-p)^t p^k, \qquad \varphi_t(u) = \exp\Big\{ -t \log \frac{1 - pe^{iu}}{1-p} \Big\}.$$

where, for $r \in \mathbb{R}$

$$\binom{r}{k} := \frac{r(r-1)\cdots(r-k+1)}{k!}, \qquad \sum_{k=0}^{\infty} \binom{r}{k} x^k = (1+x)^r.$$

**EX7a:** The compound Poisson distribution ($CPOI$): Let $\xi_1, \xi_2 \ldots$ be i.i.d. with distribution

$$\mathbf{P}\Big( \xi_j < x \Big) = G(x),$$

and $\nu \sim POI(\varrho)$ independent of the $\xi_j$-s. Then we call

$$X := \sum_{j=1}^{\nu} \xi_j \sim CPOI(G, \varrho).$$

Then:

$$CPOI(G, \varrho t) = CPOI(G, \varrho t/n)^{*n}$$

follows from infinite divisibility of Poisson: Let $N_t$ be a Poisson process of intensity $\varrho > 0$, independent of the $\xi_j$-s and

$$X_t := \sum_{j=1}^{N_t} \xi_j = \sum_{m=1}^{n} \Big( X_{mt/n} - X_{(m-1)t/n} \Big).$$

where $X_{mt/n} - X_{(m-1)t/n}$, $m = 1, 2, \ldots, n$, are i.i.d.

The characteristic function of $CPOI(G, \varrho t)$ is

$$
\begin{aligned}
\varphi_t(u) &= \mathbf{E}\Big( \exp\{iuX_t\} \Big) \\
&= \sum_{m=0}^{\infty} e^{-\varrho t} \frac{(\varrho t)^m}{m!} \Big( \int_{-\infty}^{\infty} e^{iuy} dG(y) \Big)^m \\
&= \exp \Big\{ \varrho t \int_{-\infty}^{\infty} (e^{iuy} - 1) dG(y) \Big\}.
\end{aligned}
$$

**EX7b:** The centred compound Poisson distribution ($CCPOI$):
Let $\xi_1, \xi_2 \ldots$ be i.i.d. with distribution

$$\mathbf{P}\Big(\xi_j < x\Big) = G(x),$$

and $\nu \sim POI(\varrho)$ independent of the $\xi_j$-s. Then we call

$$\widetilde{X} := \sum_{j=1}^{\nu} \xi_j - \varrho \mathbf{E}\Big(\xi_j\Big) \sim CCPOI(G, \varrho).$$

**Remark:** Mind that $CCPOI(G, \varrho) \neq CPOI(\tilde{G}, \varrho)$! (Here $\tilde{G}$ is the centred distribution.)

Then:

$$CCPOI(G, \varrho t) = CCPOI(G, \varrho t/n)^{*n}$$

follows from infinite divisibility of Poisson:

Let $N_t$ be a Poisson process of intensity $\varrho > 0$, independent of the $\xi_j$-s and

$$\widetilde{X}_t := \sum_{j=1}^{N_t} \xi_j - \varrho t \mathbf{E}\big(\,\xi\,\big) = \sum_{m=1}^{n} \Big(\widetilde{X}_{mt/n} - \widetilde{X}_{(m-1)t/n}\Big).$$

where $\widetilde{X}_{mt/n} - \widetilde{X}_{(m-1)t/n}$, $m = 1, 2, \ldots, n$, are i.i.d.

The characteristic function of $CCPOI(G, \varrho t)$ is

$$\varphi_t(u) = \mathbf{E}\Big(\, \exp\{iu\widetilde{X}_t\}\,\Big)$$

$$= \sum_{m=0}^{\infty} e^{-\varrho t} \frac{(\varrho t)^m}{m!} \Big(\int_{-\infty}^{\infty} e^{iuy} dG(y)\Big)^m \exp\Big\{ - iu\varrho t \mathbf{E}\big(\,\xi\,\big)\Big\}$$

$$= \exp\Big\{\varrho t \int_{-\infty}^{\infty} (e^{iuy} - 1 - iuy) dG(y)\Big\}.$$

**REMARKS:**

**REM1:** In all examples we have seen a one parameter family of random variables $\left(X_t\right)_{t\geq 0}$ such that for any $n \in \mathbb{N}$

$$X_t \sim \sum_{m=1}^{n} X'_{t/n,m}$$

where $X'_{t/n,1}, X'_{t/n,2}, \ldots, X'_{t/n,n} \sim X_{t/n}$ are i.i.d.

It is reasonable to expect the existence of a process $t \mapsto X_t$ with stationary and independent increments and $X_0 = 0$

= a Lévy process.

[Mind the difference between a one-parameter family of random variables and a process: process = consistent family of joint distributions of finite dimensional marginals.]

**EX1:** $t \mapsto X_t$ is standard Brownian motion.

**EX2:** The Cauchy process: Let $(\xi_s, \eta_s)$ two-dim. Brownian motion, $(\xi_0, \eta_0) = (0, 0)$. Let

$$\tau_t := \inf\{s : \xi_s = t\}, \qquad X_t := \eta_{\tau_t}.$$

**EX3:** Stable processes. E.g. for $\alpha = \frac{1}{2}, \kappa = 1$:

$$X_t := \tau_t \quad \text{of the previous example.}$$

**EX4:** $X_t$ is the Poisson process.

**EX5, EX6:** later

**EX7:** Defined from start as a process: the compound Poisson process.

**REM2:** If $X$ and $Y$ are infinitely divisible and independent then $Z := X + Y$ is infinitely divisible. (HW!)

If $X_t$ and $Y_t$ are two independent Lévy processes then $Z_t := X_t + Y_t$ is also a Lévy process. (HW!)

**REM3:**

**Lemma.** *If* $X$ *is infinitely divisible and* $\varphi(u) := \mathbf{E}\Big( \exp\{iuX\} \Big)$ *then* $(\forall u \in \mathbb{R}) : \varphi(u) \neq 0$.

*Proof.* Let $X_{n,1}, \ldots, X_{n,n}$ be i.i.d. so that $X \sim X_{n,1} + \cdots + X_{n,n}$. Then $X_{n,1} \xrightarrow{\mathbf{P}} 0$ as $n \to \infty$ (HW!). It follows that for any $u \in \mathbb{R}$

$$\lim_{n\to\infty} \varphi(u)^{1/n} = \lim_{n\to\infty} \mathbf{E}\Big( \exp\{iuX_{n,1}\} \Big) = 1. \qquad \square$$

Thus: $\psi : \mathbb{R} \to \mathbb{C}$, $\qquad \psi(u) := \log \varphi(u) \qquad$ is well defined.

**REM4:** If $t \mapsto X_t$ is a Lévy process with $X_1 \sim X$, then

$$\varphi_t(u) := \mathbf{E}\Big( \text{exp}\{iuX_t\} \Big) = \text{exp}\{t\psi(u)\}.$$

**REM5:** If $X$ is infinitely divisible and $\varphi(u) := \mathbf{E}\Big( \text{exp}\{iuX\} \Big)$ then for any $a > 0$ and $\beta > 0$:

$$\widetilde{\varphi}(u) := \Big(\varphi(au)\Big)^{\beta}$$

is infinitely divisible chf.
If $t \mapsto X_t$ is a Lévy process then so is $t \mapsto \widetilde{X}_t := aX_{\beta t}$ and

$$\widetilde{\varphi}_t(u) = \mathbf{E}\Big( \text{exp}\{iu\widetilde{X}_t\} \Big) = \mathbf{E}\Big( \text{exp}\{iuaX_{\beta t}\} \Big)$$

$$= \text{exp}\{\beta t\psi(au)\} = \Big(\varphi_t(au)\Big)^{\beta}.$$

**REM6:** If $X_n$, $n = 1, 2 \ldots$ are infinitely divisible and $X_n \Rightarrow X$ then $X$ is also infinitely divisible. (HW!)

**REM7:** If $X$ is infinitely divisible then for any $K < \infty$: $\mathbf{P}\Big( |X| > K \Big) > 0$. (HW!)

**BACK TO THE EXAMPLES:**

All the previous examples are derived in some way from the compound Poisson.

**EX1, Normal:**

$\xi_j^{(\varepsilon)}$, $j = 1, 2, \ldots$ i.i.d. with distribution

$$\mathbf{P}\Big(\xi_j^{(\varepsilon)} = \pm\varepsilon\Big) = 1/2.$$

$N_t^{(\varepsilon)}$ Poisson process of intensity $\varepsilon^{-2}$, independent of the $\xi_j^{(\varepsilon)}$-s.

*CPOI* process and its chf.:

$$X_t^{(\varepsilon)} := \sum_{j=1}^{N_t^{(\varepsilon)}} \xi_j^{(\varepsilon)}, \qquad \exp\{t\psi^{(\varepsilon)}(u)\} := \mathbf{E}\Big(\exp\{iuX_t^{(\varepsilon)}\}\Big).$$

Compute $\psi^{(\varepsilon)}(u)$ and its $\lim_{\varepsilon\to 0}$:

$$\psi^{(\varepsilon)}(u) = \varepsilon^{-2}(\cos(\varepsilon u) - 1) \to -\frac{u^2}{2}.$$

**EX2, Symmetric stable of index $\alpha \in (0,2)$:**

$\xi_j^{(\varepsilon)}$, $j = 1, 2, \ldots$ i.i.d. with symmetric distribution

$$\mathbf{P}\Big( |\xi_j^{(\varepsilon)}| > |y| \Big) = \min\{(|y|/\varepsilon)^{-\alpha}, 1\}$$

$$\frac{d}{dy}\mathbf{P}\Big( \xi_j^{(\varepsilon)} < y \Big) = \frac{1}{2}\alpha\varepsilon^{\alpha}|y|^{-\alpha-1}1\!\!1_{|y|>\varepsilon}.$$

$N_t^{(\varepsilon)}$ Poisson process of intensity $\varepsilon^{-\alpha}$, independent of the $\xi_j^{(\varepsilon)}$-s.

$CPOI$ process and its chf.:

$$X_t^{(\varepsilon)} := \sum_{j=1}^{N_t^{(\varepsilon)}} \xi_j^{(\varepsilon)}, \qquad \exp\{t\psi^{(\varepsilon)}(u)\} := \mathbf{E}\Big( \exp\{iuX_t^{(\varepsilon)}\} \Big).$$

Compute $\psi^{(\varepsilon)}(u)$ and its $\lim_{\varepsilon \to 0}$:

$$\psi^{(\varepsilon)}(u) = \varepsilon^{-\alpha} \int_{|y|>\varepsilon} (e^{iuy} - 1) \frac{1}{2} \alpha \varepsilon^{\alpha} |y|^{-\alpha-1} dy$$

$$= \alpha \int_{\varepsilon}^{\infty} (\cos(uy) - 1) y^{-\alpha-1} dy$$

$$\xrightarrow{(1)} \alpha \int_{0}^{\infty} (\cos(uy) - 1) y^{-\alpha-1} dy = -c\,|u|^{\alpha},$$

where

$$c := \alpha \int_0^{\infty} \frac{1 - \cos y}{y^{\alpha+1}} dy.$$

(1): absolute integrability at $0$ and at $\infty$.

**EX3, Skew stable:** later, **EX4, Poisson:** nothing to prove,

**EX5, Gamma:**

$\xi_j^{(\varepsilon)}$, $j = 1, 2, \ldots$ i.i.d. with distribution density

$$\frac{d}{dy}\mathbf{P}\Big(\xi_j^{(\varepsilon)} < y\Big) = \frac{1}{\varrho(\varepsilon)}\frac{e^{-y}}{y}1\!\!1_{y>\varepsilon}, \quad \text{with} \quad \varrho(\varepsilon) := \int_\varepsilon^\infty \frac{e^{-y}}{y}dy.$$

$N_t^{(\varepsilon)}$ Poisson process of intensity $\varrho(\varepsilon)$, independent of the $\xi_j^{(\varepsilon)}$-s.

$CPOI$ process and its chf.:

$$X_t^{(\varepsilon)} := \sum_{j=1}^{N_t^{(\varepsilon)}} \xi_j^{(\varepsilon)}, \qquad \exp\{t\psi^{(\varepsilon)}(u)\} := \mathbf{E}\Big( \exp\{iuX_t^{(\varepsilon)}\} \Big).$$

Compute $\psi^{(\varepsilon)}(u)$ and its $\lim_{\varepsilon \to 0}$:

$$\psi^{(\varepsilon)}(u) = \varrho(\varepsilon) \int_{\varepsilon}^{\infty} \left(e^{iuy} - 1\right) \varrho(\varepsilon)^{-1} \frac{e^{-y}}{y} dy$$

$$= \int_{\varepsilon}^{\infty} \left(e^{iuy} - 1\right) \frac{e^{-y}}{y} dy$$

$$\xrightarrow{(1)} \int_{0}^{\infty} (e^{iuy} - 1) \frac{e^{-y}}{y} dy \overset{(2)}{=} -\log(1 - iu)$$

(1): absolute integrability at 0. (2): HW!

**EX6, Negative binomial:** HW: Construct $NB(p,t)$ as compound Poisson.

**EX3a, Skew stable, $STAB(\alpha, \kappa = 1, c, 0)$, with $\alpha \in (0, 1)$:**

$\xi_j^{(\varepsilon)}$, $j = 1, 2, \ldots$ i.i.d. with distribution

$$\mathbf{P}\Big(\xi_j^{(\varepsilon)} > y\Big) = \min\{(y/\varepsilon)^{-\alpha}, 1\}, \qquad y > 0,$$

$$\frac{d}{dy}\mathbf{P}\Big(\xi_j^{(\varepsilon)} < y\Big) = \alpha\varepsilon^{\alpha}|y|^{-\alpha-1}\mathbb{1}_{y>\varepsilon}, \qquad y > 0.$$

$N_t^{(\varepsilon)}$ Poisson process of intensity $\varepsilon^{-\alpha}$, independent of the $\xi_j^{(\varepsilon)}$-s.

$CPOI$ process and its chf.:

$$X_t^{(\varepsilon)} := \sum_{j=1}^{N_t^{(\varepsilon)}} \xi_j^{(\varepsilon)}, \qquad \exp\{t\psi^{(\varepsilon)}(u)\} := \mathbf{E}\Big(\exp\{iuX_t^{(\varepsilon)}\}\Big).$$

Compute $\psi^{(\varepsilon)}(u)$ and its $\lim_{\varepsilon\to 0}$:

$$\begin{aligned}
\psi^{(\varepsilon)}(u) &= \varepsilon^{-\alpha}\int_{\varepsilon}^{\infty}(e^{iuy}-1)\alpha\varepsilon^{\alpha}y^{-\alpha-1}dy \\
&= \alpha\int_{\varepsilon}^{\infty}(e^{iuy}-1)y^{-\alpha-1}dy \\
&\overset{(1)}{\longrightarrow} \alpha\int_{0}^{\infty}(e^{iuy}-1)y^{-\alpha-1}dy \\
&\overset{(2)}{=} -\Gamma(1-\alpha)\cos\frac{\pi\alpha}{2}\left(1-i\tan\frac{\pi\alpha}{2}\operatorname{sgn}(u)\right)|u|^{\alpha}.
\end{aligned}$$

(1): The real part is absolutely integrable at $\infty$ and at 0 for any $\alpha\in(0,2)$ ✓.
The imaginary part is absolutely integrable at $\infty$ for any $\alpha\in(0,2)$ ✓, but at 0 only for $\alpha\in(0,1)$ !!!

(2) Computation on next page.

$$
\begin{aligned}
\alpha \int_0^\infty (e^{iy} - 1) y^{-\alpha-1} dy &\overset{(1)}{=} \lim_{\varepsilon\to 0} \alpha \int_0^\infty (e^{-(\varepsilon-i)y} - 1) y^{-\alpha-1} dy \\
&= -\lim_{\varepsilon\to 0} (\varepsilon - i)\alpha \int_0^\infty \Big( \int_0^y e^{-(\varepsilon-i)z} dz \Big) y^{-\alpha-1} dy \\
&\overset{(2)}{=} -\lim_{\varepsilon\to 0} (\varepsilon - i) \int_0^\infty \Big( \alpha \int_z^\infty y^{-\alpha-1} dy \Big) e^{-(\varepsilon-i)z} dz \\
&= -\lim_{\varepsilon\to 0} (\varepsilon - i) \int_0^\infty e^{-(\varepsilon-i)z} z^{-\alpha} dz \\
&\overset{(3)}{=} -\lim_{\varepsilon\to 0} (\varepsilon - i)^\alpha \int_0^\infty e^{-z} z^{-\alpha} dz \\
&= -\Gamma(1-\alpha) e^{-i\pi\alpha/2}.
\end{aligned}
$$

(1): DC, valid only for $\alpha \in (0, 1)$ !!!; (2): Fubini;

(3): Change of integration path in $\mathbb{C}$ (HW!).

**EX3b, Skew stable, $STAB(\alpha, \kappa = 1, c, 0)$, with $\alpha \in (1, 2)$:**

Centre!

$\xi_j^{(\varepsilon)}$, $j = 1, 2, \ldots$ i.i.d. with distribution

$$\mathbf{P}\Big(\xi_j^{(\varepsilon)} > y\Big) = \min\{(y/\varepsilon)^{-\alpha}, 1\}, \qquad\qquad y > 0,$$

$$\frac{d}{dy}\mathbf{P}\Big(\xi_j^{(\varepsilon)} < y\Big) = \alpha\varepsilon^{\alpha}|y|^{-\alpha-1}1\!\!1_{y>\varepsilon}, \qquad y > 0.$$

$N_t^{(\varepsilon)}$ Poisson process of intensity $\varepsilon^{-\alpha}$, independent of the $\xi_j^{(\varepsilon)}$-s.

$CCPOI$ process and its chf.:

$$\widetilde{X}_t^{(\varepsilon)} := \sum_{j=1}^{N_t^{(\varepsilon)}} \xi_j^{(\varepsilon)} - \mathbf{E}\Big(N_t^{(\varepsilon)}\Big)\mathbf{E}\Big(\xi_j^{(\varepsilon)}\Big), \quad \exp\{t\psi^{(\varepsilon)}(u)\} := \mathbf{E}\Big(\exp\{iu\widetilde{X}_t^{(\varepsilon)}\}\Big).$$

Compute $\psi^{(\varepsilon)}(u)$ and its $\lim_{\varepsilon\to 0}$:

$$\begin{aligned}\psi^{(\varepsilon)}(u) &= \varepsilon^{-\alpha}\int_{\varepsilon}^{\infty}(e^{iuy}-1-iuy)\alpha\varepsilon^{\alpha}y^{-\alpha-1}dy \\ &= \alpha\int_{\varepsilon}^{\infty}(e^{iuy}-1-iuy)y^{-\alpha-1}dy \\ &\xrightarrow{(1)} \alpha\int_{0}^{\infty}(e^{iuy}-1-iuy)y^{-\alpha-1}dy \\ &\stackrel{(2)}{=} \frac{\Gamma(2-\alpha)}{\alpha-1}\cos\frac{\pi\alpha}{2}\Big(1-i\tan\frac{\pi\alpha}{2}\,\mathrm{sgn}(u)\Big)|u|^{\alpha}.\end{aligned}$$

(1): The real part is absolutely integrable at $0$ and at $\infty$ for any $\alpha\in(0,2)$ ✓.
The centred imaginary part is absolutely integrable at $0$ for any $\alpha\in(0,2)$ ✓, but at $\infty$ only for $\alpha\in(1,2)$ !!!

(2) Computation on next page.

$$
\begin{aligned}
\alpha \int_0^\infty (e^{iy} - 1 - iy) y^{-\alpha-1} dy &= i\alpha \int_0^\infty \Big( \int_0^y (e^{iz} - 1) dz \Big) y^{-\alpha-1} dy \\
&\overset{(1)}{=} i \int_0^\infty \Big( \alpha \int_z^\infty y^{-\alpha-1} dy \Big) (e^{iz} - 1) dz \\
&= i \int_0^\infty (e^{iz} - 1) z^{-\alpha} dz \\
&\overset{(2)}{=} \frac{-1}{\alpha - 1} \int_0^\infty e^{iz} z^{1-\alpha} dz \\
&\overset{(3)}{=} \frac{i^{-\alpha}}{\alpha - 1} \int_0^\infty e^{-z} z^{1-\alpha} dz \\
&= \Gamma(2-\alpha)/(\alpha - 1) e^{-i\pi\alpha/2}.
\end{aligned}
$$

(1): Fubini; (2): integration by parts;

(3): Change of integration path in $\mathbb{C}$ (HW!).

**Definition.** *A non-negative sigma-finite measure on $\mathbb{R}$ for which*

$$(1): \qquad \mu((-\infty,-1]\cup[1,\infty))<\infty,$$

$$(2): \qquad \int y^2 1\!\!1_{|y|<1}\,d\mu(y)<\infty,$$

$$(3): \qquad \mu(0)=0,$$

*is called Lévy measure.*

**Theorem** (Aleksandr Yakovlevich Khinchin, Paul Lévy). *Characteristic functions of infinitely divisible distributions are exactly the functions of the form $\varphi_t(u)=\exp\{t\psi(u)\}$ with*

$$\psi(u)=ibu-\frac{\sigma^2}{2}u^2+\int_{\mathbb{R}}\Big(e^{iuy}-1-iuy1\!\!1_{|y|<1}\Big)d\mu(y), \qquad \text{(LH)}$$

*where $b\in\mathbb{R}$, $\sigma^2\geq 0$, and $\mu$ is a Lévy measure.*

**Remarks:**

**REM1:** The parameters $b$, $\sigma^2$ and $\mu$ are uniquely determined.

**REM2:** (LH) is the Lévy-Khinchin formula.

**REM3:** If $g : \mathbb{R} \to \mathbb{R}$ is such that

$$\int_{\mathbb{R}} \left| y \mathbb{1}_{|y|<1} - g(y) \right| d\mu(y) < \infty$$

then the Lévy-Khinchin formula can be written

$$\psi(u) = ib'u - \frac{\sigma^2}{2} u^2 + \int_{\mathbb{R}} \Big( e^{iuy} - 1 - iug(y) \Big) d\mu(y),$$

with

$$b' = b - \int_{\mathbb{R}} \Big( y \mathbb{1}_{|y|<1} - g(y) \Big) d\mu(y)$$

**REM4:** Another usual conventional choice

$$\psi(u) = ibu - \frac{\sigma^2}{2}u^2 + \int_{\mathbb{R}} \Big(e^{iuy} - 1 - iu\frac{y}{1+y^2}\Big) d\mu(y),$$

**REM5:** If

$$\int_{\mathbb{R}} |y| d\mu(y) < \infty$$

then the Lévy-Khinchin formula can be written

$$\psi(u) = ibu - \frac{\sigma^2}{2}u^2 + \int_{\mathbb{R}} \Big(e^{iuy} - 1\Big) d\mu(y),$$

*Proof.*
We prove that functions $\varphi_t(u) = \exp\{t\psi(u)\}$ with $\psi(u)$ given by (LH) are indeed chf-s of infinitely divisible distributions. We write $\psi(u) = \psi_1(u) + \psi_2(u) + \psi_2(u) + \psi_4(u)$, with

$$\psi_1(u) = ibu, \qquad \psi_2(u) = -\frac{\sigma^2}{2}u^2$$

$$\psi_3(u) = \int_{\mathbb{R}} \left(e^{iuy} - 1\right) 1\!\!1_{|y|\geq 1} d\mu(y)$$

$$\psi_4(u) = \int_{\mathbb{R}} \left(e^{iuy} - 1 - iuy\right) 1\!\!1_{|y|<1} d\mu(y)$$

Then:

– $\psi_1(u)$ comes from a simple shift by $b$.

– $\psi_2(u)$ comes from a Gaussian $\sim N(0, \sigma^2)$.

– $\psi_3(u)$ comes from a compound Poisson $CPOI(\varrho, F)$ with

$$\varrho = \mu(\{y : |y| \geq 1\}), \qquad dF(y) = \varrho^{-1} 1\!\!1_{|y|\geq 1} d\mu(y).$$

– $\psi_4(u)$ comes as weak limit of a sequence of $CCPOI(\varrho^{(\varepsilon)}, G^{(\varepsilon)})$:

$$\int_{\mathbb{R}} \left(e^{iuy} - 1 - iuy\right) 1\!\!1_{|y|<1} d\mu(y) =$$

$$= \lim_{\varepsilon \to 0} \int_{\mathbb{R}} \left(e^{iuy} - 1 - iuy\right) 1\!\!1_{\varepsilon<|y|<1} d\mu(y)$$

$$= \lim_{\varepsilon \to 0} \varrho^{(\varepsilon)} \int_{\mathbb{R}} \left(e^{iuy} - 1 - iuy\right) 1\!\!1_{\varepsilon<|y|<1} dG^{(\varepsilon)}(y)$$

where

$$\varrho^{(\varepsilon)} := \mu(\{y : \varepsilon < |y| < 1\}), \qquad dG^{(\varepsilon)}(y) = (\varrho^{(\varepsilon)})^{-1} 1\!\!1_{\varepsilon<|y|<1} d\mu(y).$$

□

## LÉVY MEASURE OF STABLE LAWS:

For $STAB(\alpha, \kappa, c, b)$

$$\psi(u) = ibu + c\int_{-\infty}^{\infty}\Big(e^{iuy} - 1 - iuy\mathbb{1}_{|y|<1}\Big)d\mu_{\alpha,\kappa}(y),$$

$$d\mu_{\alpha,\kappa}(y) := \Big(\frac{1+\kappa}{2}\mathbb{1}_{y>0} + \frac{1-\kappa}{2}\mathbb{1}_{y<0}\Big)\frac{1}{|y|^{\alpha+1}}dy.$$

**Remarks:**

**REM1:** These are the only homogeneous Lévy measures.

**REM2:** Alternative forms:

$$\alpha \in (0,1): \qquad \psi(u) = ib'u + c\int_{-\infty}^{\infty}\Big(e^{iuy} - 1\Big)d\mu_{\alpha,\kappa}(y),$$

$$\alpha \in (1,2): \qquad \psi(u) = ib'u + c\int_{-\infty}^{\infty}\Big(e^{iuy} - 1 - iuy\Big)d\mu_{\alpha,\kappa}(y),$$

*Proof.* If

$$\psi(u) = ibu - \frac{\sigma^2}{2}u^2 + \int_{-\infty}^{\infty} \Big(e^{iuy} - 1 - iuy\mathbb{1}_{|y|<1}\Big) d\mu(y),$$

and $a > 0$ then

$$\tilde{\psi}(u) := \psi(au) = i\tilde{b}u - \frac{\tilde{\sigma}^2}{2}u^2 + \int_{-\infty}^{\infty} \Big(e^{iuy} - 1 - iuy\mathbb{1}_{|y|<1}\Big) d\tilde{\mu}(y),$$

with

$$\tilde{b} = ab - a\int y\Big(\mathbb{1}_{|y|<1} - \mathbb{1}_{|y|<a^{-1}}\Big) d\mu(y), \quad \tilde{\sigma}^2 = a^2\sigma^2, \quad d\tilde{\mu}(y) = d\mu(y/a).$$

Stability: $(\forall\ a_1, a_2 > 0)$ $(\exists\ a_3 > 0, b_3 \in \mathbb{R})$ such that:

$$\psi(a_1 u) + \psi(a_2 u) = ib_3 u + \psi(a_3 u).$$

It follows that

$$(1) \qquad d\mu(y/a_1) + d\mu(y/a_2) = d\mu(y/a_3),$$

$$(2) \qquad (a_1^2 + a_2^2)\sigma^2 = a_3^2\sigma^2.$$

(1) implies homogeneity of $\mu$:

with some $\alpha \in (0,2)$ and $C_+, C_- \geq 0$

$$d\mu(y) = \left(C_+ \mathbb{1}_{y>0} + C_- \mathbb{1}_{y<0}\right)\frac{1}{|y|^{\alpha+1}}dy,$$

and

$$\left\{ C_+ + C_- > 0 \right\} \quad \Longrightarrow \quad \left\{ a_1^\alpha + a_2^\alpha = a_3^\alpha . \right\}.$$

From (2) it follows that either $\sigma^2 = 0$ or $C_+ = 0 = C_-$. $\square$

## POISSON POINT PROCESSES:

Let $(S, d)$ be a complete, separable, locally compact metric space. E.g. $S = \mathbb{R}^n$, or $S = (-\infty, 0) \cup (0, \infty)$ with properly chosen metrization.

The space of locally finite point systems:

$$\Pi = \Pi(S) := \left\{ \mathcal{X} \subset S : (\forall K \Subset S) : |\mathcal{X} \cap K| < \infty \right\}.$$

$\Pi(S)$ is endowed with a natural metric topology, . . . , Borel sigma algebra $\mathcal{F}$.

Counting functions: for $K \Subset S$

$$m_K : \Pi \to \mathbb{N}, \qquad m_K(\mathcal{X}) := |\mathcal{X} \cap K|.$$

**Definition.** *A (random) point process on* $(S, d)$ *is a* $(\Pi, \mathcal{F})$*-valued random variable,* $\Xi$*. I.e. it is a probability measure on* $(\Pi, \mathcal{F})$*.*

**Definition.** *Let* $\mu$ *be a sigma-finite, tight positive measure on* $S$*. The Poisson point process with intensity mesure* $\mu$ *– denoted* $PPP(\mu)$ *– is* the unique *point process* $\Xi$ *on* $(S, d)$ *satisfying the following: If* $K_1, \ldots, K_n \Subset S$ *are disjoint then* $\left(m_{K_1}(\Xi), \ldots, m_{K_n}(\Xi)\right)$ *are independent, and* $m_{K_j}(\Xi) \sim POI(\mu(K_i))$*.*

**Remarks:**

Existence of $PPP(\mu)$: see constructions on next page.

Uniqueness of $PPP(\mu)$:

**CONSTRUCTION FOR $\mu(S) < \infty$:**

Let $\xi_1, \xi_2, \dots \in S$ be i.i.d., and $\nu \in \mathbb{N}$ independent of the $\xi_j$-s, with distribution

$$\mathbf{P}\Big(\xi_j \in A\Big) = \frac{\mu(A)}{\mu(S)}, \qquad \nu \sim POI(\mu(S)).$$

Then $\Xi := \big\{\xi_1, \xi_2, \dots, \xi_\nu\big\}$ is $PPP(\mu)$. (HW!).

**CONSTRUCTION FOR $\mu(S) = \infty$:**

Let $S = \cup_{k=1}^{\infty} S_k$, with disjoint $S_k$-s and $(\forall k) : \mu(S_k) < \infty$.

Let $\mu_k(\cdot) := \mu(\cdot \cap S_k)$, and $\Xi_k \sim PPP(\mu_k)$ as defined above.

Then $\Xi := \bigcup_{k=1}^{\infty} \Xi_k$ is $PPP(\mu)$. (HW!).

**REM1:** A theorem of Rényi:

**Theorem.** *Let* $\mu$ *be a non-atomic measure on* $S$. *(That is:* $(\forall x \in S) : \mu(\{x\}) = 0$.*) Let* $\mathcal{A} \subset \mathcal{P}(S)$ *generate the Borel-algebra of* $(S, d)$. *If for a point process* $\mathcal{X}$ *the following holds:*

$$(\forall A \in \mathcal{A}, \quad \textit{with } \mu(A) < \infty) : \qquad m_A(\mathcal{X}) \sim POI(\mu(A)),$$

*then* $\mathcal{X} \sim PPP(\mu)$.

HW: Give counterexample with atomic $\mu$!

**REM2:** Relation to $CPOI(\varrho, G)$: For $S = \mathbb{R}$, $\varrho := \mu(S) < \infty$, $dG(y) := \varrho^{-1} d\mu(y)$:

$$\sum_{\xi \in \Xi} \xi =: X \sim CPOI(\varrho, G).$$

**REM3:** Relation to Lévy measure, Lévy-Khinchin formula – summable case: $S = (0, \infty)$.

If

$$(1): \quad \int_1^\infty d\mu(y) < \infty, \qquad (2): \quad \int_0^1 y d\mu(y) < \infty,$$

let

$$X_1 := \sum_{\xi \in \Xi \cap [1,\infty)} \xi, \qquad X_2 := \sum_{\xi \in \Xi \cap (0,1)} \xi,$$

Then

$$\mathbf{P}\Big( X_1 < \infty \Big) = 1 \qquad (\checkmark),$$

$$\mathbf{E}\Big( X_2 \Big) = \int_0^1 y d\mu(y) < \infty. \qquad \text{(HW!)}$$

The characteristic function of $X := X_1 + X_2$ is:

$$\mathbf{E}\Big( \exp\{iuX\} \Big) = \exp\Big\{ \int_0^\infty (e^{iuy} - 1) d\mu(y) \Big\}.$$

**REM4:** Relation to Lévy measure, Lévy-Khinchin formula – non-summable case: $S = (0, \infty)$.

If

$$(1): \quad \int_1^\infty d\mu(y) < \infty, \qquad (2): \quad \int_0^1 y^2 d\mu(y) < \infty,$$

let

$$X_1 := \sum_{\xi \in \Xi \cap [1,\infty)} \xi, \qquad X_{2,\varepsilon} := \sum_{\xi \in \Xi \cap (\varepsilon,1)} \xi - \int_\varepsilon^1 y d\mu(y).$$

Then

$$\mathbf{P}\Big( X_1 < \infty \Big) = 1 \qquad (\checkmark),$$

$$\mathbf{P}\Big( \exists \lim_{\varepsilon \to 0} X_{2,\varepsilon} =: X_2 \Big) = 1, \qquad (HW!)$$

(Hint: Compute $\mathbf{Var}\Big( X_{2,\varepsilon} \Big)$ and use Kolmogorov's criterion.)

The characteristic function is of $X := X_1 + X_2$ is:

$$\mathbf{E}\Big( \exp\{iuX\} \Big) = \exp\Big\{ \int_0^\infty (e^{iuy} - 1 - iuy\mathbb{1}_{0<y<1}) d\mu(y) \Big\}.$$

## BACK TO STABLE CONVERGENCE:

**Definition.** *The function* $L:(0,\infty)\to(0,\infty)$ *is slowly varying (at infinity) iff*

$$(\forall a>0): \qquad \lim_{x\to\infty}\frac{L(ax)}{L(x)}=1.$$

**Examples, remarks, HWs:**

(1) If $\lim_{x\to\infty}L(x)=b\in(0,\infty)$ then obviously $L$ is s.v.

(2) For any $\beta\in\mathbb{R}$, $L(x):=(\log x)^{\beta}$ is s.v.

(3) Show that for $\beta<1$ and $c\in\mathbb{R}$, $L(x):=\exp\{c(\log x)^{\beta}\}$ is s.v.

(4) Construct a s.v. function $L$ for which

$$\liminf_{x\to\infty}L(x)=0, \quad \limsup_{x\to\infty}L(x)=\infty.$$

**Definition.** *The function* $U : (0, \infty) \to (0, \infty)$ *is regularly varying (at infinity) iff*

$$(\forall a > 0) : \qquad \lim_{x\to\infty} \frac{U(ax)}{U(x)} \qquad \text{exists}$$

**Fact:** The function $x \mapsto U(x)$ is regularly varying at $\infty$ if and only if $U(x) = x^{\beta} L(x)$ with some $\beta \in \mathbb{R}$ and $L(x)$ slowly varying. (HW!)

**Some basic facts about slowly varying functions:**

**(1)** If $\beta > -1$ then

$$\int_1^x y^\beta L(y)dy = \quad \Big(\frac{1}{\beta+1} + o(1)\Big)x^{\beta+1}L(x).$$

**(2)** If $\beta < -1$ then

$$\int_x^\infty y^\beta L(y)dy = -\Big(\frac{1}{\beta+1} + o(1)\Big)x^{\beta+1}L(x).$$

**(3)**

$$L(x) = a(x)\exp\Big\{\int_1^x \frac{\epsilon(y)}{y}dy\Big\}$$

where $\quad \exists \quad \lim_{y\to\infty} a(y) =: c, \qquad \lim_{y\to\infty} \epsilon(y) = 0.$

**Theorem** (Skew stable limit theorem.). *Let* $\xi_1, \xi_2, \ldots$ *be i.i.d. random variables. Assume*

(1) $\mathbf{P}\Big( |\xi_1| > x \Big) = x^{-\alpha} L(x)$ *with* $\alpha \in (0,1) \cup (1,2)$,

(2) $\exists \quad \lim_{x \to +\infty} \frac{\mathbf{P}\Big( \xi_1 > x \Big)}{\mathbf{P}\Big( \xi_1 < -x \Big)} =: \frac{1+\kappa}{1-\kappa} \in [0, \infty]$.

*Define*

$$a_n := \inf\{x \, : \, \mathbf{P}\Big( |\xi_1| > x \Big) < n^{-1}\} = n^{1/\alpha} \widetilde{L}(n).$$

*(i)* $\alpha \in (0,1)$ *case:*

$$\mathbf{E}\Big( \exp\{iuS_n/a_n\} \Big) \to \exp\Big\{ \int_{\mathbb{R}} \Big(e^{iuy} - 1\Big) d\mu_{\alpha,\kappa}(y) \Big\}.$$

*(ii)* $\alpha \in (1,2)$ *case:*

$$\mathbf{E}\Big( \exp\{iu(S_n - n\mathbf{E}(\xi_1))/a_n\} \Big) \to \exp\Big\{ \int_{\mathbb{R}} \Big(e^{iuy} - 1 - iuy\Big) d\mu_{\alpha,\kappa}(y) \Big\}.$$

*Proof:*

**Lemma** (1).

$$\left\{\frac{\xi_1}{a_n}, \frac{\xi_2}{a_n}, \dots, \frac{\xi_n}{a_n}\right\} =: \Xi_n \Rightarrow PPP(\mu_{\alpha,\kappa}).$$

**Lemma** (2). *Let* $X_n$, $n = 1, 2, \dots$, *be a sequence of random variables and assume that for any* $r = 1, 2, \dots$, $X_n$ *is decomposed as* $X_n = Y_{n,r} + Z_{n,r}$. *If*

$$Y_{n,r} \overset{n\to\infty}{\Longrightarrow} Y_{\infty,r} \overset{r\to\infty}{\Longrightarrow} Y,$$

*and*

$$(\forall \delta > 0): \qquad \lim_{r\to\infty} \limsup_{n\to\infty} \mathbf{P}\Big( |Z_{n,r}| \geq \delta \Big) = 0,$$

*Then* $X_n \overset{n\to\infty}{\Longrightarrow} Y$.

**Case $\alpha \in (0,1)$:**

$$X_n := \frac{S_n}{a_n} = \sum_{j=1}^{n} \frac{\xi_j}{a_n}$$

$$= \sum_{j=1}^{n} \frac{\xi_j}{a_n} \mathbb{1}_{\frac{|\xi_j|}{a_n} \geq r^{-1}} + \sum_{j=1}^{n} \frac{\xi_j}{a_n} \mathbb{1}_{\frac{|\xi_j|}{a_n} < r^{-1}}$$

$$=: Y_{n,r} + Z_{n,r}.$$

$$\mathbf{E}\left( e^{iuY_{n,r}} \right) \xrightarrow{n\to\infty} \exp\left\{ \int \left( e^{iuy} - 1 \right) \mathbb{1}_{|y| \geq r^{-1}} d\mu_{\alpha,\kappa}(y) \right\}$$

$$\xrightarrow{r\to\infty} \exp\left\{ \int \left( e^{iuy} - 1 \right) d\mu_{\alpha,\kappa}(y) \right\}$$

$$
\begin{aligned}
\mathbf{E}\Big(\,|Z_{n,r}|\,\Big) &\le n a_n^{-1}\mathbf{E}\Big(\,|\xi_1|\,\mathbb{1}_{|\xi_1|<a_n/r}\,\Big) \\
&= n a_n^{-1}\int_0^{a_n/r} x d\mathbf{P}\Big(\,|\xi_1|<x\,\Big) \\
&= n a_n^{-1}\int_0^{a_n/r} \Big\{\mathbf{P}\Big(\,|\xi_1|>x\,\Big)-\mathbf{P}\Big(\,|\xi_1|>a_n/r\,\Big)\Big\}dx \\
&= n a_n^{-1}\Big\{\int_0^{a_n/r} x^{-\alpha}L(x)dx-(a_n/r)^{1-\alpha}L(a_n/r)\Big\} \\
&= n a_n^{-1}(a_n/r)^{1-\alpha}L(a_n/r)\Big(\frac{\alpha}{1-\alpha}+o(1)\Big) \\
&= \Big\{n a_n^{-\alpha}L(a_n)\Big\}\,\frac{L(a_n/r)}{L(a_n)}\,\Big(\frac{\alpha}{1-\alpha}+o(1)\Big)\,r^{\alpha-1}.
\end{aligned}
$$

Hence: $\lim_{r\to\infty}\limsup_{n\to\infty}\mathbf{E}\Big(\,|Z_{n,r}|\,\Big)=0.$

**Case** $\alpha \in (1,2)$**:**

$$
\begin{aligned}
X_n := \frac{S_n - n\mathbf{E}(\xi_1)}{a_n} &= \sum_{j=1}^{n} \Big\{ \frac{\xi_j}{a_n} - \mathbf{E}\Big( \frac{\xi_j}{a_n} \Big) \Big\} \\
&= \quad \sum_{j=1}^{n} \Big\{ \frac{\xi_j}{a_n} 1\!\!1_{\frac{|\xi_j|}{a_n} \geq r^{-1}} - \mathbf{E}\Big( \frac{\xi_j}{a_n} 1\!\!1_{\frac{|\xi_j|}{a_n} \geq r^{-1}} \Big) \Big\} \\
&\quad + \sum_{j=1}^{n} \Big\{ \frac{\xi_j}{a_n} 1\!\!1_{\frac{|\xi_j|}{a_n} < r^{-1}} - \mathbf{E}\Big( \frac{\xi_j}{a_n} 1\!\!1_{\frac{|\xi_j|}{a_n} < r^{-1}} \Big) \Big\} \\
&=: Y_{n,r} + Z_{n,r}.
\end{aligned}
$$

$$Y_{n,r} = \sum_{j=1}^{n} \frac{\xi_j}{a_n} 1\!\!1_{\frac{|\xi_j|}{a_n} \geq r^{-1}} - n a_n^{-1} \mathbf{E}\Big( \xi_1 1\!\!1_{\frac{\xi_1}{a_n} \geq r^{-1}} \Big)$$

$$- n a_n^{-1} \mathbf{E}\Big( \xi_1 1\!\!1_{\frac{\xi_1}{a_n} \leq -r^{-1}} \Big),$$

$$\mathbf{E}\Big( e^{iuY_{n,r}} \Big) \xrightarrow{n \to \infty} \exp\Big\{ \int \Big( e^{iuy} - 1 - iuy \Big) 1\!\!1_{|y| \geq r^{-1}} d\mu_{\alpha,\kappa}(y) \Big\}$$

$$\xrightarrow{r \to \infty} \exp\Big\{ \int \Big( e^{iuy} - 1 - iuy \Big) d\mu_{\alpha,\kappa}(y) \Big\}$$

Computation on next page.

$$
\begin{aligned}
na_n^{-1}\mathbf{E}\Big(\xi_1 1\!\!1_{\frac{\xi_1}{a_n}\geq r^{-1}}\Big) &= na_n^{-1}\int_{a_n/r}^{\infty} x d\mathbf{P}\Big(\xi_1 < x\Big) \\
&= na_n^{-1}\int_{a_n/r}^{\infty}\Big\{-d\Big(x\mathbf{P}\Big(\xi_1 \geq x\Big)\Big) + \mathbf{P}\Big(\xi_1 \geq x\Big)dx\Big\} \\
&= na_n^{-1}\Big\{(a_n/r)^{1-\alpha}L^+(a_n/r) + \int_{a_n/r}^{\infty} x^{-\alpha}L^+(x)dx\Big\} \\
&= na_n^{-\alpha}L(a_n)\frac{L^+(a_n/r)}{L(a_n)}\Big(\frac{\alpha}{\alpha-1} + o(1)\Big)r^{\alpha-1} \\
&\overset{n\to\infty}{\longrightarrow} \frac{\kappa+1}{2}\frac{\alpha}{\alpha-1}r^{\alpha-1} \\
&= \frac{\kappa+1}{2}\int_{1/r}^{\infty} y\frac{\alpha}{y^{\alpha+1}}dy \qquad (\checkmark).
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{E}\Big( Z_{n,r}^2 \Big) &\le n a_n^{-2} \Big\{ \mathbf{E}\Big( \xi_1^2 \mathbb{1}_{|\xi_1| < a_n/r} \Big) - \mathbf{E}\Big( \xi_1 \mathbb{1}_{|\xi_1| < a_n/r} \Big)^2 \Big\} \\
&= n a_n^{-2} \Big\{ \int_0^{a_n/r} x^2 d\mathbf{P}\Big( |\xi_1| < x \Big) - \Big( \int_0^{a_n/r} x d\mathbf{P}\Big( |\xi_1| < x \Big)\Big)^2 \Big\} \\
&\le n a_n^{-2} \int_0^{a_n/r} x^2 d\mathbf{P}\Big( |\xi_1| < x \Big) \\
&= n a_n^{-2} \Big\{ -(a_n/r)^2 \mathbf{P}\Big( |\xi_1| \ge a_n/r \Big) + \int_0^{a_n/r} 2x \mathbf{P}\Big( |\xi_1| \ge x \Big) dx \Big\} \\
&= n a_n^{-2} \Big\{ -(a_n/r)^{2-\alpha} L(a_n/r) + \Big( \frac{2}{2-\alpha} + o(1) \Big) (a_n/r)^{2-\alpha} L(a_n/r) \\
&= \Big\{ n a_n^{-\alpha} L(a_n) \Big\} \frac{L(a_n/r)}{L(a_n)} \Big( \frac{\alpha}{2-\alpha} + o(1) \Big) r^{\alpha-2}.
\end{aligned}
$$

Hence: $\lim_{r\to\infty} \limsup_{n\to\infty} \mathbf{E}\Big( Z_{n,r}^2 \Big) = 0.$