Exam - June 14, 2024, Limit thms. of probab.

Family name ______________________________ **Given name** ______________________________

Signature ______________________________ **Neptun Code** ______________________________

Info: Each of the 4 questions is worth 25 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises (and sub-exercises) with a horizontal line. No calculators or electronic devices are allowed. No formula sheets are allowed. You have 100 minutes to complete this exam. You need to collect at least 40 points on this exam in order to pass the course.

1. Let us consider a r.v. X with optimistic geometric distribution: $X \sim \text{GEO}(p)$:

$$\mathbb{P}(X = k) = (1-p)^{k-1}p, \qquad k = 1, 2, 3, \ldots$$

   (a) Find the moment generating function $Z(\lambda)$ of X. For which values of λ do we have $Z(\lambda) < +\infty$?

   (b) Find the logarithmic moment generating function $\widehat{I}(\lambda)$ of X.

   (c) Show that $\widehat{I}'(\lambda) = \frac{1}{1-e^{\lambda}(1-p)}$.

   (d) Use Legendre transform to show that the large deviation rate function $I(x)$ is

$$I(x) = \begin{cases} (x-1)\ln\left(\frac{x-1}{1-p}\right) - x\ln(x) - \ln(p) & \text{if } x \geq 1, \\ +\infty & \text{if } x < 1. \end{cases}$$

   (e) Show that the exponentially tilted r.v. $X^{(\mu)} \sim \text{GEO}(p')$ for some $p' = p'(p, \mu)$ and that any $p' \in (0,1)$ can be obtained by choosing μ from the domain of $Z(\cdot)$ appropriately.

2. Let (X_n) denote a simple symmetric random walk on $\mathbb{Z}$. Let π_n denote amount of time the walker spends on the positive half-line up to time n. Let $u(k) = \mathbb{P}(X_k = 0)$. Show that

$$\mathbb{P}(\pi_{2n} = 2k) = u(2k) \cdot u(2 \cdot (n-k)), \qquad k = 0, 1, \ldots, n$$

   Instruction: You may use without proof that $\mathbb{P}(\pi_{2n} = 0) = u(2n)$.

3. Let $X_1, X_2, \ldots$ be i.i.d. random variables with p.d.f.

$$f(x) = \frac{1}{|x|^3}\mathbb{1}[\,|x| \geq 1\,].$$

   Show that

$$\frac{X_1 + \cdots + X_n}{\sqrt{n \cdot \ln(n)}} \Rightarrow \mathcal{N}(0,1)$$

   Instruction: Truncation: let $\alpha_n := \sqrt{n}\ln(\ln(n))$ and $\xi_{n,k} := X_k \cdot \mathbb{1}[\,|X_k| < \alpha_n\,]$.

4. Let X be a random variable and denote by $\varphi(t) := \mathbb{E}(e^{itX})$ ($t \in \mathbb{R}$) its characteristic function. Let us assume that $-X \sim X$, i.e., we assume that the distribution of X is symmetric.

   (a) For any $u \in \mathbb{R}_+$ let $f_u(t) := \frac{u^3t^2e^{-ut}}{2}$. Show that $\int_0^\infty \frac{1-\cos(tx)}{t^2} f_u(t)\,\mathrm{d}t = \frac{1}{2}\frac{x^2u^2}{x^2+u^2}$

   (b) Use the monotone convergence theorem to show that $\lim_{u\to\infty} \int_0^\infty \frac{1-\varphi(t)}{t^2} f_u(t)\,\mathrm{d}t = \frac{1}{2}\mathbb{E}(X^2)$.

   (c) Show that if $\limsup_{t\to 0} \big(1-\varphi(t)\big)/t^2 < +\infty$ then $\mathbb{E}(X^2) < +\infty$.

   (d) Show that if $\mathbb{E}(X^2) < +\infty$ then $\lim_{t\to 0} \big(1-\varphi(t)\big)/t^2 = \frac{1}{2}\mathbb{E}(X^2)$. *Hint:* Dominated convergence.