

THM: IF $\varphi_n(t) = \mathbb{E}(e^{itX_n})$ AND

$\forall t \in \mathbb{R}: \lim_{n \rightarrow \infty} \varphi_n(t) = \varphi(t)$, MOREOVER

φ IS CONTINUOUS AT $t=0$ THEN φ
IS THE CHAR. FN. OF SOME R.V. X

AND $X_n \Rightarrow X$ (SEE PAGE 91)

NOTE: IF $X_n \sim \mathcal{N}(0, n^2)$ THEN

$$\varphi_n(t) = e^{-n^2 t^2 / 2} \xrightarrow[n \rightarrow \infty]{} \begin{cases} 1 & \text{IF } t=0 \\ 0 & \text{IF } t \neq 0 \end{cases}$$

SO φ_n CONVERGES $\forall t \in \mathbb{R}$, BUT THE LIMIT
FUNCTION IS NOT CONTINUOUS AT $t=0$
AND INDEED X_n DOES NOT CONVERGE
WEAKLY AS $n \rightarrow \infty$: MASS ESCAPES TO ∞

HOW TO GUARANTEE TIGHTNESS OF $(X_n)_{n=1}^\infty$?

LEMMA: (LÉVY): IF $\varphi(t) = \mathbb{E}(e^{itX})$ THEN

$$\mathbb{P}(|X| \geq k) \leq \frac{k}{2} \int_{-2/k}^{2/k} (1 - \varphi(t)) dt$$

PROOF:

FUBINI

$$\frac{\kappa}{2} \int_{-2/\kappa}^{2/\kappa} (1 - \mathbb{E}(e^{itX})) dt \stackrel{A}{=} \mathbb{E} \left(\frac{\kappa}{2} \int_{-2/\kappa}^{2/\kappa} (1 - e^{itX}) dt \right) \stackrel{B}{=}$$

$$\mathbb{E} \left(2 - \frac{\kappa}{2} \int_{-2/\kappa}^{2/\kappa} e^{itX} dt \right) \stackrel{C}{=} \mathbb{E} \left(2 - \frac{\kappa}{2} \cdot \left[\frac{e^{itX}}{i \cdot X} \right]_{-2/\kappa}^{2/\kappa} \right) \stackrel{D}{=}$$

$$\mathbb{E} \left(2 - \frac{\kappa}{2} \cdot \frac{e^{i \cdot 2X/\kappa} - e^{-i \cdot 2X/\kappa}}{i \cdot X} \right) \stackrel{E}{=} \mathbb{E} \left(2 - \frac{\kappa}{2} \cdot \frac{2i \cdot \sin\left(\frac{2X}{\kappa}\right)}{i \cdot X} \right)$$

$$\stackrel{F}{=} 2 \cdot \mathbb{E} \left(1 - \frac{\sin(2X/\kappa)}{2X/\kappa} \right) \stackrel{G}{\geq}$$

NOTE:

$$\frac{\sin(x)}{x} \leq 1$$

$$2 \cdot \mathbb{E} \left(\left(1 - \frac{\sin(2X/\kappa)}{2X/\kappa} \right) \cdot \mathbb{1}[|X| \geq \kappa] \right) \geq$$

$$\stackrel{H}{\geq} 2 \cdot \mathbb{E} \left(\left(1 - \frac{\kappa}{2} \right) \cdot \mathbb{1}[|X| \geq \kappa] \right) \stackrel{I}{\geq}$$

NOTE:

$$\frac{\sin(x)}{x} \leq \frac{1}{|x|}$$

$$2 \cdot \mathbb{E} \left(\frac{1}{2} \cdot \mathbb{1}[|X| \geq \kappa] \right) \stackrel{J}{=}$$

$$= \mathbb{P}(|X| \geq \kappa) \checkmark$$

PROOF OF THM FROM PAGE 108:

FIRST WE SHOW THAT $(X_n)_{n=1}^{\infty}$ IS

A TIGHT SEQUENCE: GIVEN SOME

$\varepsilon > 0$, WANT $\tilde{K} < +\infty$ SUCH THAT

$$\mathbb{P}(|X_n| \geq \tilde{K}) \leq \varepsilon \quad \text{FOR ALL } n \in \mathbb{N}.$$

φ IS CONTINUOUS AT $t=0$ AND $\varphi(0)=1$

$$\text{THUS } \exists K : |1 - \varphi(t)| \leq \frac{\varepsilon}{4} \quad \text{IF } t \in \left[-\frac{2}{K}, \frac{2}{K}\right]$$

$$\text{THUS } \frac{K}{2} \int_{-2/K}^{2/K} (1 - \varphi(t)) dt \leq \frac{K}{2} \cdot \frac{4}{K} \cdot \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$$

$$\boxed{\varphi_n(t) \rightarrow \varphi(t) \quad \forall t \in \mathbb{R}}$$
 THUS BY DOMINATED CONV:

$$\lim_{n \rightarrow \infty} \frac{K}{2} \int_{-2/K}^{2/K} (1 - \varphi_n(t)) dt = \frac{K}{2} \int_{-2/K}^{2/K} (1 - \varphi(t)) dt \leq \frac{\varepsilon}{2}$$

$$\text{THUS } \exists n_0 : \forall n \geq n_0 : \mathbb{P}(|X_n| \geq K) \leq \varepsilon$$

$$\text{THUS } \exists \tilde{K} \geq K : \forall n \in \mathbb{N} :$$

TIGHTNESS ✓

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THUS BY HELLY'S THM THERE IS

A SUBSEQUENCE $(X_{m_{k_l}})_{l=1}^{\infty}$ SUCH THAT

$$X_{m_{k_l}} \xRightarrow{A} X \text{ FOR SOME } X. \quad \boxed{\psi(t) := \mathbb{E}(e^{itX})} \quad B$$

$$\text{THEN } \varphi_{m_{k_l}}(t) \xrightarrow[\infty]{k_l} \psi(t) \quad C \quad (\text{SEE PAGE 91})$$

$$\text{ALSO } \varphi_{m_{k_l}}(t) \xrightarrow[\infty]{k_l} \psi(t) \quad D \quad \text{THUS } \boxed{\psi \equiv \varphi} \quad E$$

THUS φ IS INDEED THE CHAR. FN OF A R.V.!

IT REMAINS TO SHOW THAT $X_m \xRightarrow{F} X$

PROOF BY CONTRADICTION: IF THERE IS A BOUNDED, CONTINUOUS $g: \mathbb{R} \rightarrow \mathbb{R}$ SUCH THAT $\mathbb{E}(g(X_m))$ DOES NOT CONVERGE TO $\mathbb{E}(g(X))$

THEN THERE IS A SUBSEQ. $(X_{m_{k_l}})$ SUCH THAT

$$\mathbb{E}(g(X_{m_{k_l}})) \xrightarrow[\infty]{k_l} C \neq \mathbb{E}(g(X)) \quad G$$

WE CHOOSE A SUB-SUB-SEQUENCE $(X_{m_{k_{l_l}}})$

S.T. $X_{m_{k_{l_l}}} \xRightarrow{H} \tilde{X}$ AS $l \rightarrow \infty$ BY HELLY.

BY THE PREVIOUS ARGUMENT:

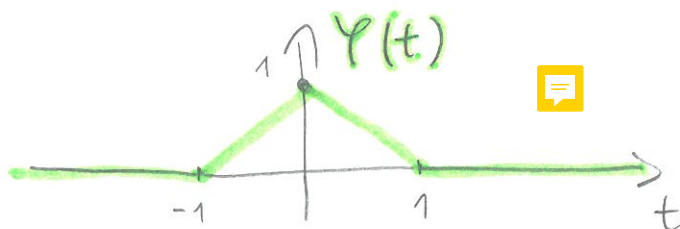
$$\varphi_{\tilde{X}}(t) \stackrel{A}{=} \varphi_X(t) \quad \text{THUS} \quad \tilde{X} \stackrel{B}{\sim} X, \text{ THUS}$$

$$X_{m_{n\ell}} \stackrel{C}{\Rightarrow} X, \text{ THUS} \quad E(g(X_{m_{n\ell}})) \xrightarrow[\infty]{L} E(g(X)) \quad \downarrow$$

EX: SHOW THAT $f(x) = \frac{1 - \cos(x)}{\pi \cdot x^2}$ IS A P.D.F.

AND SHOW THAT ITS CHAR. FUNCTION IS

$$\varphi(t) = \cos(1 - |t|)$$



SOLUTION: LET $g(x) := \cos(1 - |x|)$

WE KNOW FROM HW 6.2 (b) THAT

$$\int_{-\infty}^{\infty} e^{itx} \cdot g(x) dx = \left(\frac{\sin(t/2)}{t/2} \right)^2 = 2 \cdot \frac{1 - \cos(t)}{t^2} = \varphi(t)$$

NOW IS CONTINUOUS AND

$$\int_{-\infty}^{\infty} |\varphi(t)| dt = \int_{-\infty}^{\infty} \varphi(t) dt \leq 2 + 2 \cdot \int_1^{\infty} \frac{1}{t^2} dt \leq$$

$$\leq 2 + 4 \cdot \int_1^{\infty} \frac{1}{t^2} dt < +\infty$$

THUS BY CLAIM ON PAGE 104: 

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \psi(t) dt \stackrel{A}{=} g(x), \text{ THUS } $$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \cdot \psi(x) dx \stackrel{B}{=} g(t), \text{ THUS}$$

$$ \int_{-\infty}^{\infty} e^{itx} \cdot \frac{1 - \cos(x)}{\pi \cdot x^2} dx \stackrel{C}{=} g(-t) = g(t) = O_{\sqrt{1-|t|}} $$

IN PARTICULAR:
(t=0)

$$\int_{-\infty}^{\infty} \frac{1 - \cos(x)}{\pi \cdot x^2} dx \stackrel{D}{=} 1 \quad \checkmark $$

CLAIM: IF $\psi(t) = \mathbb{E}(e^{itX})$ THEN

$$\mathbb{E}(|X|) \stackrel{E}{=} 2 \cdot \int_0^{\infty} \frac{1 - \operatorname{Re}(\psi(t))}{\pi \cdot t^2} dt   $$

(WHERE $\psi(t) = \operatorname{Re}(\psi(t)) + i \cdot \operatorname{Im}(\psi(t))$)

PROOF: $\psi(t) \stackrel{F}{=} \mathbb{E}(\cos(tX)) + i \cdot \mathbb{E}(\sin(tX)) $

$$2 \cdot \int_0^{\infty} \frac{1 - \operatorname{Re}(\psi(t))}{\pi t^2} dt \stackrel{G}{=} \int_{-\infty}^{\infty} \frac{1 - \mathbb{E}(\cos(t \cdot |X|))}{\pi \cdot t^2} dt = $$

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FUBINI

$$\stackrel{\text{A}}{=} \mathbb{E} \left(\int_{-\infty}^{\infty} \frac{1 - \cos(t \cdot |X|)}{\pi \cdot t^2} dt \right) \stackrel{\text{B}}{=}$$

$$\begin{aligned} r &= t \cdot |X| \\ t &= r / |X| \\ dt &= \frac{1}{|X|} dr \end{aligned}$$

$$= \mathbb{E} \left(\int_{-\infty}^{\infty} \frac{1 - \cos(r)}{\pi \cdot (r/|X|)^2} \cdot \frac{1}{|X|} dr \right) \stackrel{\text{C}}{=}$$

$$= \mathbb{E} \left(|X| \cdot \underbrace{\int_{-\infty}^{\infty} \frac{1 - \cos(r)}{\pi \cdot r^2} dr}_{\substack{\text{D} \\ 1}} \right) = \mathbb{E}(|X|)$$

IT WAS OK TO USE FUBINI EVEN

IF $\mathbb{E}(|X|) = +\infty$ BECAUSE IF

$g: \mathbb{R}^2 \rightarrow \mathbb{R}_+$ (I.E., IF $g(x, t) \geq 0$), THEN

$$\mathbb{E} \left(\int_{-\infty}^{\infty} g(x, t) dt \right) = \int_{-\infty}^{\infty} \mathbb{E}(g(x, t)) dt$$

AND IF ONE SIDE IS $+\infty$ THEN

THE OTHER SIDE IS $+\infty$.  