THM (ERDŐS-KAC, 1940)

LET $\mathbb{P}(U_n = k) \overset{A}{=} \frac{1}{n}$, $k \in \{1, 2, \ldots, n\}$

LET Z_n DENOTE THE NUMBER OF PRIME DIVISORS OF U_n. THEN

$$\frac{Z_n - \ln(\ln(n))}{\sqrt{\ln(\ln(n))}} \overset{B}{\Rightarrow} N(0,1)$$

PROOF: DENOTE BY $\mathbb{P}$ THE SET OF PRIMES

CLAIM:

$$\sum_{\substack{p \in \mathbb{P} \\ p \le n}} \frac{1}{p} \overset{C}{=} \ln(\ln(n)) + O(1)$$

PROOF: (WE WILL ONLY PROVE THE $\geq$ PART)

$$\ln(n) \overset{D}{=} \int_1^n \frac{1}{x}\,dx \overset{E}{\le} \sum_{k=1}^{n} \frac{1}{k} \overset{F}{\le} \prod_{\substack{p \in \mathbb{P} \\ p \le n}} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \ldots\right)$$

$$\overset{G}{=} \prod_{\substack{p \in \mathbb{P} \\ p \le n}} \frac{1}{1 - \frac{1}{p}}, \text{ THUS } \ln(\ln(n)) \overset{H}{\le} \sum_{\substack{p \in \mathbb{P} \\ p \le n}} -\ln\left(1 - \frac{1}{p}\right) \overset{I}{=}$$

$$\overset{J}{=} \sum_{\substack{p \in \mathbb{P} \\ p \le n}} -\left(-\frac{1}{p} + O\left(\frac{1}{p^2}\right)\right) \overset{K}{=} \sum_{\substack{p \in \mathbb{P} \\ p \le n}} \frac{1}{p} + O(1) \checkmark$$

BACK TO PROOF OF ERDŐS-KAC:

LET $Y_{n,p} :=_{A} \mathbb{1}[\, p \text{ DIVIDES } U_n \,]$

THEN $Z_n =_{B} \sum_{\substack{p \in \mathcal{P} \\ p \le n}} Y_{n,p}$

$\mathbb{E}(Y_{n,p}) =_{C} \mathbb{P}(p \text{ DIVIDES } U_n) =_{D} \frac{1}{n} \cdot \left\lfloor \frac{n}{p} \right\rfloor \approx \frac{1}{p}$

(NOTE: $(Y_{n,p})_{p \in \mathcal{P}}$ ARE NOT INDEPENDENT)

TRICK: TRUNCATION. LET US DEFINE

$T_n =_{E} \sum_{\substack{p \in \mathcal{P} \\ p \le \alpha_n}} Y_{n,p}$

WHERE

$\alpha_n :=_{F} n^{1/\ln(\ln(n))}$

ENOUGH TO SHOW:

$(\star) \rightarrow \frac{T_n - \ln(\ln(n))}{\sqrt{\ln(\ln(n))}} \Rightarrow_{G} \mathcal{N}(0,1)$

BECAUSE: $\ln(\alpha_n) =_{H} \ln(n) / \ln(\ln(n))$

$\ln(\ln(\alpha_n)) =_{I} \ln(\ln(n)) - \ln(\ln(\ln(n)))$

THUS $T_n \le Z_n$ AND

$\mathbb{E}(Z_n - T_n) \overset{J}{\le} \sum_{\alpha_n < p \le n} \frac{1}{p} \overset{K}{=} \ln(\ln(\ln(n))) + O(1)$

BY CLAIM FROM PAGE 129

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THUS $\frac{Z_n - T_n}{\sqrt{\ln(\ln(n))}} \xrightarrow{A} 0$ BY MARKOV'S INEQ.

HENCE IF WE PROVE (*) THEN (:) WILL FOLLOW.

LET $(X_p)_{p \in \mathbb{P}}$ BE INDEP., $X_p \overset{B}{\sim} \mathrm{BER}\left(\frac{1}{p}\right)$

LET $S_n := \sum_{\substack{p \in \mathbb{P} \\ p \le \alpha_n}} X_p$ (C)

LEMMA: $\frac{S_n - \ln(\ln(n))}{\sqrt{\ln(\ln(n))}} \xrightarrow{D} N(0,1)$

PROOF. ENOUGH TO SHOW:

$$\frac{S_n - \ln(\ln(\alpha_n))}{\sqrt{\ln(\ln(\alpha_n))}} \xrightarrow{E} N(0,1) \quad \leftarrow (:)'$$

SINCE $\ln(\ln(\alpha_n)) \overset{F}{=} \ln(\ln(n)) - \ln(\ln(\ln(n)))$

WE WILL USE LINDEBERG TO PROVE (:)'.

$$\mathbb{E}(S_n) \overset{G}{=} \sum_{\substack{p \in \mathbb{P} \\ p \le \alpha_n}} \frac{1}{p} \overset{H}{=} \ln(\ln(\alpha_n)) + O(1)$$

$$\mathrm{Var}(S_n) \overset{I}{=} \sum_{\substack{p \in \mathbb{P} \\ p \le \alpha_n}} \left(\frac{1}{p} - \frac{1}{p^2}\right) \overset{J}{=} \ln(\ln(\alpha_n)) + O(1)$$

LINDEBERG'S CONDITION:

LET $\tilde{X}_p := X_p - \mathbb{E}(X_p)$ (A) THEN $|\tilde{X}_p| \leq 1$ (B)

GIVEN SOME $\varepsilon > 0$:

$$\lim_{n\to\infty} \frac{1}{\ln(\ln(\alpha_n))} \sum_{\substack{p \in \mathbb{P} \\ p \leq \alpha_n}} \mathbb{E}\left(|\tilde{X}_p|^2 \cdot \mathbb{1}\left[|\tilde{X}_p| > \varepsilon \cdot \sqrt{\ln(\ln(\alpha_n))}\right]\right) = 0 \quad (C)$$

SINCE $\varepsilon \cdot \sqrt{\ln(\ln(\alpha_n))} \geq 1$ (D) IF n IS LARGE ENOUGH

LET $S_n^* := \frac{S_n - \ln(\ln(n))}{\sqrt{\ln(\ln(n))}}$ (E) $\quad T_n^* = \frac{T_n - \ln(\ln(n))}{\sqrt{\ln(\ln(n))}}$ (F)

KNOW: $S_n^* \Rightarrow N(0,1)$ (G) $\quad$ WANT: $T_n^* \Rightarrow N(0,1)$ (H)

ACTUALLY WE KNOW MORE (HW 9.1):

$\forall k \in \mathbb{N}: \lim_{n\to\infty} \mathbb{E}\left((S_n^*)^k\right) \overset{(I)}{=} \mathbb{E}(X^k)$, $X \sim N(0,1)$

WE WILL NOW SHOW:

$\forall k \in \mathbb{N}: \lim_{n\to\infty} \mathbb{E}\left((T_n^*)^k\right) \overset{(J)}{=} \mathbb{E}(X^k)$, -II-

LEMMA: $\left|\mathbb{E}(S_n^r) - \mathbb{E}(T_n^r)\right| \underset{A}{\leq} \frac{1}{n} \cdot (\alpha_n)^r$

PROOF: LET $\boxed{N \underset{B}{:=} \sum_{\substack{p \in \mathcal{P} \\ p \leq \alpha_n}} 1}$ $\boxed{\text{NOTE: } N \underset{C}{\leq} \alpha_n}$

$$(X_1 + \ldots + X_N)^r \underset{D}{=} \underbrace{\sum_{\ell=1}^{N} \sum_{\substack{r_1, \ldots, r_\ell \geq 1 \\ r_1 + \ldots + r_\ell = r}} \sum_{1 \leq m_1 < m_2 < \ldots < m_\ell \leq N} \overbrace{C(\ell; r_1, \ldots, r_\ell)}^{\geq 0} \cdot X_{m_1}^{r_1} \cdot \ldots \cdot X_{m_\ell}^{r_\ell}}$$

THUS: $\mathbb{E}(S_n^r) \underset{E}{=} \text{—''—} \cdot \mathbb{E}(X_{m_1}^{r_1} \cdot \ldots \cdot X_{m_\ell}^{r_\ell})$

NOW IF $1 < p_1 < p_2 < \ldots < p_\ell \leq \alpha_n$ THEN

$$\mathbb{E}(X_{p_1}^{r_1} \cdot \ldots \cdot X_{p_\ell}^{r_\ell}) \underset{F}{=} \mathbb{E}(X_{p_1} \cdot \ldots \cdot X_{p_\ell}) \underset{G}{=} \frac{1}{p_1 \cdot p_2 \cdot \ldots \cdot p_\ell}$$

$$\mathbb{E}(Y_{n,p_1}^{r_1} \cdot \ldots \cdot Y_{n,p_\ell}^{r_\ell}) \underset{H}{=} \mathbb{E}(Y_{n,p_1} \cdot \ldots \cdot Y_{n,p_\ell}) \underset{I}{=} \frac{1}{n} \cdot \left\lfloor \frac{n}{p_1 \cdot p_2 \cdot \ldots \cdot p_\ell} \right\rfloor$$

THUS:

$$0 \underset{J}{\leq} \mathbb{E}(S_n^r) - \mathbb{E}(T_n^r) \underset{K}{\leq} \sum_{\ell=1}^{N} \sum_{r_1, \ldots, r_\ell} \sum_{\ldots} C(\ell; r_1, \ldots, r_\ell) \cdot \frac{1}{n} \underset{L}{=}$$

$$= \frac{1}{n} \cdot (1 + 1 + \ldots + 1)^r = \frac{1}{n} \cdot N^r \underset{M}{\leq} \frac{1}{n} \cdot (\alpha_n)^r$$

✓

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COROLLARY:

$$\forall r \in \mathbb{N}: \quad \left| \mathbb{E}\left[(S_n^*)^r \right] - \mathbb{E}\left[(T_n^*)^r \right] \right| \xrightarrow[n \to \infty]{} 0 \qquad \text{A}$$

PROOF:

$$\mathbb{E}\left[(S_n^*)^r \right] \overset{\text{B}}{=} \frac{1}{\ln(\ln(n))^{r/2}} \mathbb{E}\left[\left(S_n - \ln(\ln(n)) \right)^r \right]$$

$$\overset{\text{C}}{=} \frac{1}{\ln(\ln(n))^{r/2}} \sum_{l=0}^{r} \binom{r}{l} \cdot \mathbb{E}\left(S_n^l \right) \cdot \left(-\ln(\ln(n)) \right)^{r-l}$$

THUS

$$\left| \mathbb{E}\left((S_n^*)^r \right) - \mathbb{E}\left((T_n^*)^r \right) \right| \overset{\text{D}}{=}$$

$$\left| \frac{1}{\ln(\ln(n))^{r/2}} \sum_{l=0}^{r} \binom{r}{l} \cdot \left(\mathbb{E}\left(S_n^l \right) - \mathbb{E}\left(T_n^l \right) \right) \cdot \left(-\ln(\ln(n)) \right)^{r-l} \right|$$

$$\overset{\text{E}}{\leq} \sum_{l=0}^{r} \binom{r}{l} \cdot \frac{1}{n} \cdot d_n^l \cdot \ln(\ln(n))^{r-l} \overset{\text{F}}{=}$$

$$= \frac{1}{n} \cdot \left(d_n \cdot \ln(\ln(n)) \right)^r \xrightarrow[n \to \infty]{\text{G}} 0$$

BECAUSE $d_n = n^{1/\ln(\ln(n))}$, THUS $\forall \varepsilon > 0$

$d_n \overset{\text{H}}{\leq} n^{\varepsilon}$ IF n IS LARGE ENOUGH.

THUS

$$\forall k \in \mathbb{N}: \boxed{\lim_{n\to\infty} \mathbb{E}\left((T_n^*)^k\right) = \mathbb{E}(X^k)} \quad , \quad X \sim \mathcal{N}(0,1)$$

A B

IT REMAINS TO SHOW THAT THIS IMPLIES

$$\boxed{T_n^* \Rightarrow \mathcal{N}(0,1)}$$

C

THE MOMENT PROBLEM:

IF $\forall k \in \mathbb{N}: \mathbb{E}(V^k) = \mathbb{E}(W^k) < +\infty$

D E

DOES THIS IMPLY THAT $\boxed{V \sim W}$?

F

UNFORTUNATELY NOT, E.G. THE LOG-NORMAL DISTRIBUTION IS NOT DETERMINED BY ITS MOMENTS (HW 9.2). HOWEVER:

LEMMA: IF $M_k = \mathbb{E}(V^k) = \mathbb{E}(W^k)$ AND

G H

$$\limsup_{n\to\infty} \left(\frac{|M_k|}{k!}\right)^{1/k} =: R^{-1} < +\infty$$

I J

THEN $\boxed{V \sim W}$.

K

PROOF: SEE NEXT PAGE

PROOF: $\varphi(t) \overset{A}{=} \mathbb{E}(e^{it\cdot V}) \overset{B}{=} \sum_{k=0}^{\infty} \frac{M_k}{k!} i^k \cdot t^k$

THIS POWER SERIES WILL HAVE A RADIUS OF CONVERGENCE $R > 0$,

THUS $\varphi(t)$ UNIQUELY EXTENDS AS AN ANALYTIC FUNCTION TO THE STRIP $\{t \in \mathbb{C} : \mathrm{Im}(t) \in (-R, R)\}$, SEE PAGE 85.

THUS $\boxed{\varphi_V(t) \overset{C}{\equiv} \varphi_W(t)}$, THUS $\boxed{V \overset{D}{\sim} W}$

EX: $\mathcal{N}(0,1)$: $M(\lambda) \overset{E}{=} e^{\lambda^2/2} \overset{F}{=} \sum_{\ell=0}^{\infty} \frac{(\lambda^2/2)^\ell}{\ell!}$

THUS $\sum_{k=0}^{\infty} \frac{M_k}{k!} \lambda^k \overset{G}{=} \sum_{\ell=0}^{\infty} \frac{\lambda^{2\ell}}{2^\ell \cdot \ell!}$, THUS

$M_k = 0$ IF k IS ODD AND

IF $k = 2\ell$ THEN $M_k = M_{2\ell} \overset{H}{=} \frac{(2\ell)!}{2^\ell \cdot \ell!}$

RADIUS OF CONVERGENCE: $R = +\infty$

THM: LET $M_{n,k} \overset{A}{:=} \mathbb{E}\left((V_n)^k\right) < +\infty$

IF $\forall k \in \mathbb{N}$: $\boxed{\lim_{n\to\infty} M_{n,k} \overset{B}{=:} M_k}$ AND

THE SEQUENCE OF MOMENTS $(M_k)_{k=0}^{\infty}$

UNIQUELY DETERMINES A DISTRIBUTION (RANDOM VARIABLE): V, THEN $\boxed{V_n \overset{C}{\Rightarrow} V}$

PROOF: $(V_n)_{n=1}^{\infty}$ IS A TIGHT SEQUENCE:

$$\mathbb{P}(|V_n| \geq K) \overset{D}{=} \mathbb{P}(V_n^2 \geq K^2) \overset{E}{\leq} \frac{M_{n,2}}{K^2}$$

(E: MARKOV)

THUS GIVEN SOME $\varepsilon > 0$ WE HAVE

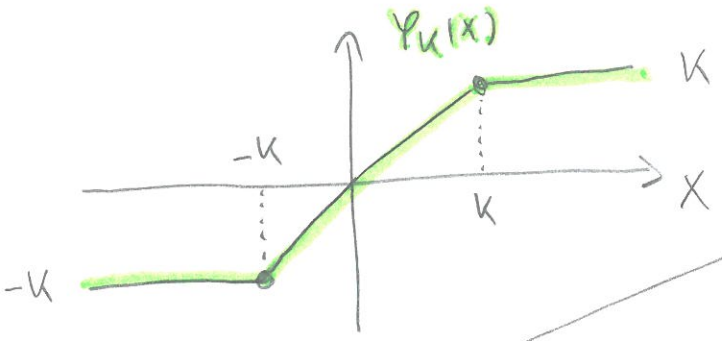
$\sup_n \mathbb{P}(|V_n| \geq K) \overset{F}{\leq} \varepsilon$ IF $K \overset{G}{:=} \sqrt{\frac{\sup_n M_{n,2}}{\varepsilon}}$

NOW WE SHOW THAT IF V_{n_l} IS A SUBSEQUENCE WHICH CONVERGES IN DISTRIBUTION $(V_{n_l} \overset{H}{\Rightarrow} \tilde{V})$ THEN

$\boxed{\mathbb{E}(\tilde{V}^k) \overset{I}{=} M_k}$, $k \in \mathbb{N}$

INDEED: LET US DEFINE

$$\varphi_K(x) \overset{A}{=} x \cdot \mathbb{1}[|x| \le K] + \operatorname{sign}(x) \cdot K \cdot \mathbb{1}[|x| > K]$$



IF q IS EVEN: BY MONOTONE CONV.

IF q IS ODD: DOMINATED CONV.

$$\mathbb{E}(\tilde{V}^q) \overset{B}{=} \lim_{K\to\infty} \mathbb{E}\left(\varphi_K(\tilde{V})^q\right)$$

SINCE $V_{m'} \Rightarrow \tilde{V}$, φ_K IS BOUNDED CONTINUOUS

$$\overset{C}{=} \lim_{K\to\infty} \lim_{m'\to\infty} \mathbb{E}\left(\varphi_K(V_{m'})^q\right)$$

$$\overset{D}{=} \lim_{K\to\infty} \lim_{m'\to\infty} \left(\mathbb{E}(V_{m'}^q) - \mathbb{E}\left(V_{m'}^q - \varphi_K(V_{m'})^q\right)\right)$$

$$\overset{E}{=} \underbrace{\lim_{m'\to\infty} M_{m',q}}_{M_q\ (F)} - \underbrace{\lim_{K\to\infty} \lim_{m'\to\infty} \mathbb{E}\left(V_{m'}^q - \varphi_K(V_{m'})^q\right)}_{G}$$

G: ☹ ← WE WILL SHOW THAT THIS IS ZERO

$$\left|\mathbb{E}\left(V_m^q - \varphi_K(V_m)^q\right)\right| \overset{H}{\le} \mathbb{E}\left(|V_m|^q \cdot \mathbb{1}[|V_m| \ge K]\right)$$

$$\overset{I}{\le} \sqrt{\mathbb{E}\left(V_m^{2q}\right)} \cdot \sqrt{\mathbb{E}\left(\mathbb{1}[|V_m| \ge K]^2\right)} \overset{J}{=}$$

CAUCHY-SCHWARZ

$$\sqrt{M_{m,2q}} \cdot \sqrt{\mathbb{P}(|V_m| \ge K)} \overset{K}{=} ☺$$

$$\text{☺} \overset{A}{\leq} \sqrt{M_{m,2R}} \cdot \sqrt{M_{m,2}/K^2} \overset{B}{\leq} \frac{\sqrt{\sup_m M_{m,2R} \cdot M_{m,2}}}{K}$$

↑ A — MARKOV INEQ

THUS $\text{☹} \overset{C}{\leq} \lim_{K\to\infty} \frac{\sqrt{\sup_m M_{m,2R} \cdot M_{m,2}}}{K} \overset{D}{=} 0$ ✓

THUS $\mathbb{E}(\tilde{V}^R) \overset{E}{=} M_R$, $\forall R \in \mathbb{N}$, THUS $\boxed{\tilde{V} \overset{F}{\sim} V}$

TO SHOW THAT THE WHOLE SEQUENCE V_m CONVERGES IN DISTRIBUTION TO V, JUST REPEAT THE ARGUMENT FROM PAGE 110 – 112.