

ASTROPHYSICS: (HOLTS MARK'S PROBLEM)

STARS ARE SCATTERED IN THE UNIVERSE IN A UNIFORM FASHION.

WHAT IS THE TOTAL GRAVITATIONAL FORCE AT THE CENTER OF THE UNIV. GENERATED BY ALL OF THE STARS?

1-DIMENSIONAL MODEL: $n \in \mathbb{N}$,

$\underbrace{X_{n,1}, \dots, X_{n,n}}_A$ i.i.d. $\text{UNIF}[-n, n]$ B

POSITIONS OF STARS.

NOTE: DENSITY OF STARS IS $\frac{1}{2}$

A STAR LOCATED AT $x \in \mathbb{R}$ GENERATES AT THE ORIGIN THE FORCE

$\text{sgn}(x) \cdot |x|^{-p}$, THUS: TOTAL FORCE:

$$R_n = \sum_{i=1}^n \text{sgn}(X_{n,i}) \cdot |X_{n,i}|^{-p}$$

QUESTION:

$$R_n \Rightarrow (?)$$

AS $n \rightarrow \infty$?

THM:

(A) IF $\frac{1}{2} < p < +\infty$ THEN
 $\lim_{n \rightarrow \infty} E(e^{it \cdot R_n}) \underset{A}{=} e^{-c \cdot |t|^{\frac{1}{p}}}$ WITH

$$C = \lambda \cdot \int_0^{\infty} \frac{1 - \omega(y)}{y^{\lambda+1}} dy, \quad \lambda = \frac{1}{p}$$

(B) IF $p = \frac{1}{2}$ THEN $\frac{R_n}{\sqrt{\ln(n)}} \underset{B}{\Rightarrow} N(0, 1)$

(C) IF $0 \leq p < \frac{1}{2}$: $\frac{R_n}{n^{\frac{1}{2}-p}} \underset{C}{\Rightarrow} N(0, \sigma^2)$

$$\sigma^2 = \frac{1}{1-2p}$$

REMARK: SO IF $\frac{1}{2} < p$ THEN AN INFINITE UNIVERSE IS OK, BUT IF $p \leq \frac{1}{2}$ THEN R_n DIVERGES AS $n \rightarrow \infty$, SO AN INFINITE UNIVERSE IS IMPOSSIBLE IN THIS CASE.

PROOF: LET $Y_1, Y_2, \dots$ I.I.D. $\text{UNI}[-1, 1]$

$$\frac{\sum_{n=1}^n \text{sgn}(Y_n) \cdot |Y_n|^{-p}}{n^p} \underset{A}{=} \sum_{n=1}^n \text{sgn}(n \cdot Y_n) \cdot |n \cdot Y_n|^{-p} \underset{B}{\sim} R_n$$

$$Z_n \underset{C}{:=} \text{sgn}(Y_n) \cdot |Y_n|^{-p}$$

$$S_n \underset{D}{:=} \sum_{n=1}^n Y_n$$

THUS $\frac{S_n}{n^p} \underset{E}{\sim} R_n$

$$-Z_n \underset{F}{\sim} Z_n \text{ is SYMM.}$$

IF $x \geq 0$ THEN $P(Z_n > x) \underset{G}{=} P(0 < Y_n < x^{-1/p})$

THUS $P(Z_n > x) \underset{H}{=} \begin{cases} 1/2 & \text{IF } 0 \leq x \leq 1 \\ \frac{1}{2} x^{-1/p} & \text{IF } x \geq 1 \end{cases}$

THUS IF $\alpha = 1/p$ THEN BY PAGE 148-150

WE HAVE $\lim_{n \rightarrow \infty} E(e^{it R_n}) \underset{J}{=} e^{-c \cdot |t|^\alpha}$

$$p > \frac{1}{2} \iff 0 < \alpha < \frac{1}{2}$$

THUS (A) HOLDS ✓

(B) $\lambda = \frac{1}{P} = 2$ P.D.F. OF Z_n :

$f(x) = \frac{1}{|x|^3} \cdot \mathbb{I}[|x| > 1]$, THUS BY

PAGE
121-122

$\frac{S_n}{\sqrt{n \cdot \ln(n)}} = \frac{R_n}{\sqrt{\ln(n)}} \Rightarrow N(0, 1)$

BORDERLINE
CASE
OF C.L.T.

(C) $E(Z_n) = 0$, $\sigma^2 = \text{Var}(Z_n) = E(Z_n^2) =$

$= E(|Y_n|^{-2p}) = \int_{-1}^1 |y|^{-2p} \cdot \frac{1}{2} dy = \frac{1}{1-2p} < +\infty$

THUS BY CENTRAL LIMIT THM:

$0 \leq p < \frac{1}{2}$

$\frac{S_n}{\sqrt{n} \cdot \sigma} = \frac{R_n \cdot n^p}{n^{1/2} \cdot \sigma} = \frac{R_n}{n^{1/2-p} \cdot \sigma} \Rightarrow N(0, 1)$

THUS $\frac{R_n}{n^{1/2-p}} \Rightarrow N(0, \sigma)$ ✓



EX: d -DIMENSIONAL HOLTSMARK:

STARS ARE LOCATED IN $\mathbb{R}^d$ ACCORDING TO A HOMOGENEOUS POISSON POINT PROCESS (PPP) OF DENSITY ς .

DEF: PPP(ς) ON $\mathbb{R}^d$: DENOTE BY X_A THE NUMBER OF STARS IN $A \subseteq \mathbb{R}^d$ THEN $X_A \sim \text{POI}(\varsigma \cdot |A|)$ WHERE $|A|$ DENOTES THE d -DIM. VOLUME OF A .
MOREOVER IF $A_1, A_2, \dots, A_k$ ARE DISJOINT SUBSETS OF $\mathbb{R}^d$, THEN $X_{A_1}, X_{A_2}, \dots, X_{A_k}$ ARE INDEPENDENT.

DENOTE BY $\underline{X}_1^{(\varsigma)}, \underline{X}_2^{(\varsigma)}, \dots$ THE LOCATIONS OF STARS IN SOME (ARBITRARY) ORDERING.
A STAR LOCATED AT $\underline{x} \in \mathbb{R}^d$ GENERATES AT THE ORIGIN THE FORCE $\vec{F}(\underline{x})$, WHERE

$$\vec{F}(\underline{x}) = \|\underline{x}\|^{-p} \cdot \frac{\underline{x}}{\|\underline{x}\|}$$

$$\vec{R}^{(\varsigma)} := \sum_i \vec{F}(\underline{X}_i^{(\varsigma)})$$

WHAT IS THE DISTRIBUTION OF $\vec{R}^{(s)}$?
 (IF THE SUM DEFINING $\vec{R}^{(s)}$ IS "CONVERGENT")

HERE IS A FORMAL CALCULATION
 THAT SHOWS THAT $\vec{R}^{(s)}$ HAS TO BE
 A "d-DIM. SYMM. STABLE DISTRIBUTION
 OF INDEX $\alpha = \frac{d}{p}$ " LET'S SEE:

① $\vec{R}^{(s)} \sim s^{p/d} \cdot \vec{R}^{(1)}$ (SCALING)

$(\vec{X}_1^{(s)}, \vec{X}_2^{(s)}, \dots) \sim (\xi^{-1/d} \cdot \vec{X}_1^{(1)}, \xi^{-1/d} \cdot \vec{X}_2^{(1)}, \dots)$

PPP(s)

PPP(1) SCALED BY $\xi^{-1/d}$ IS PPP(s)

THUS $\vec{R}^{(s)} = \sum_i \vec{F}(\vec{X}_i^{(s)}) \sim \sum_i \vec{F}(\xi^{-1/d} \cdot \vec{X}_i^{(1)}) =$

$= \sum_i \xi^{p/d} \cdot \vec{F}(\vec{X}_i^{(1)}) \sim \xi^{p/d} \cdot \vec{R}^{(1)}$ ✓

② "INFINITE DIVISIBILITY":

THE UNION OF n INDEPENDENT COPIES OF $PPP(1)$ IS A $PPP(n)$

THUS THE SUM OF n INDEPENDENT COPIES OF $\vec{R}^{(1)}$ HAS THE SAME DISTRIBUTION AS $\vec{R}^{(n)}$ (BY ②)

$$\text{THUS } \underbrace{\vec{R}_1^{(1)} + \dots + \vec{R}_n^{(1)}}_{P/d} \underset{A}{\sim} \vec{R}^{(1)} \quad (\text{BY ①})$$

THIS MATCHES THE DEFINITION OF SYMMETRIC STABLE DISTRIBUTION

OF INDEX $\alpha = \frac{d}{p}$ (SEE PAGE 146)

THIS ONLY MAKES SENSE IF $\alpha < 2$

BUT LUCKILY IN REAL WORLD:

$$\alpha = 3, \quad p = 2, \quad \alpha = \frac{3}{2}$$

HOLTS MARK DISTRIBUTION: SYMM. STABLE

DISTRIBUTION OF INDEX $\alpha = \frac{3}{2}$ [PAGE 162]

REMARK: BUT THE UNIVERSE CANNOT BE INFINITE, BECAUSE THE NIGHT SKY WOULD BE INFINITELY BRIGHT:

$$\text{IF } A_n := B(R+1) \setminus B(R) \\ = \{ \underline{x} \in \mathbb{R}^d : R \leq \| \underline{x} \| \leq R+1 \}$$

$\sum_{A_n}$ = NUMBER OF STARS IN A_n

$$|A_n| \lesssim R^2 \quad (\text{SINCE } d=3)$$

THUS $\sum_{A_n} \lesssim R^2$. IF EACH STAR

EMITS A UNIT AMOUNT OF LIGHT, THE TOTAL AMOUNT OF LIGHT THAT REACHES US FROM A_n IS

$$\text{ROUGHLY } \sum_{A_n} \cdot R^{-2} \lesssim R^2 \cdot R^{-2} = 1$$

THUS THE TOTAL AMOUNT OF LIGHT THAT REACHES US FROM THE WHOLE UNIVERSE IS ROUGHLY $1+1+1+\dots = +\infty$

