

EX: $X \sim \mathcal{N}(m, \sigma)$ NORMAL DISTRIBUTION

DENSITY: $f_{m, \sigma}(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right)$

MOMENT GENERATING FUNCTION OF X ?

FIRST: $Y \sim \mathcal{N}(0, 1)$

$$Z_Y(\lambda) = E(e^{\lambda \cdot Y}) = \int_{-\infty}^{\infty} e^{\lambda x} \cdot f_{0,1}(x) dx =$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\lambda x - \frac{x^2}{2}\right) dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-\lambda)^2}{2} + \frac{\lambda^2}{2}\right) dx$$

$$= e^{\lambda^2/2} \int_{-\infty}^{\infty} f_{\lambda,1}(x) dx = e^{\lambda^2/2} \quad \text{NOW } X = m + \sigma \cdot Y$$

$$Z_X(\lambda) = E(e^{\lambda \cdot X}) = E(e^{\lambda \cdot m} \cdot e^{\lambda \cdot \sigma \cdot Y}) =$$

$$e^{\lambda \cdot m} \cdot E(e^{\lambda \cdot \sigma \cdot Y}) = e^{\lambda \cdot m} \cdot Z_Y(\lambda \cdot \sigma) = e^{\lambda \cdot m + \frac{\sigma^2 \cdot \lambda^2}{2}}$$

LOG. MOM GEN. F'N OF X : $\hat{I}(\lambda) = \lambda \cdot m + \frac{\sigma^2 \cdot \lambda^2}{2}$

LEGENDRE TRANSFORM OF \hat{I} :

$$I(x) = \max_{\lambda} \{ \lambda \cdot x - \hat{I}(\lambda) \}$$

$$\frac{d}{d\lambda} (\lambda \cdot x - \hat{I}(\lambda)) = x - m - \lambda \cdot \sigma^2 = 0 \Rightarrow$$

$$\lambda^* = \frac{x-m}{\sigma^2}$$

$$I(x) = \lambda^* \cdot x - \hat{I}(\lambda^*) = \frac{x-m}{\sigma^2} \cdot x - \frac{x-m}{\sigma^2} \cdot m - \frac{\sigma^2 \cdot \left(\frac{x-m}{\sigma^2}\right)^2}{2} =$$
$$= \frac{(x-m)^2}{2\sigma^2}$$

THUS IF X_1, X_2, \dots I.I.D. $N(m, \sigma)$

$S_m = X_1 + \dots + X_m$ AND $X > m$ THEN

$$P\left(\frac{S_m}{n} \geq x\right) \leq \exp(-n \cdot I(x)) = \exp\left(-n \cdot \frac{(x-m)^2}{2\sigma^2}\right)$$

(SEE PAGE 8, 9)

IS THIS UPPER BOUND SHARP? ☆

NOTE: $S_m \sim N(n \cdot m, \sigma \cdot \sqrt{n})$ $\frac{S_m}{n} \sim N\left(m, \frac{\sigma}{\sqrt{n}}\right)$

$$\text{THUS } P\left(\frac{S_m}{n} \geq x\right) = \int_x^{\infty} \frac{\sqrt{n}}{\sqrt{2\pi} \cdot \sigma} \cdot \exp\left(-n \cdot \frac{(y-m)^2}{2\sigma^2}\right) dy$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log\left(P\left(\frac{S_m}{n} \geq x\right)\right) =$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log\left(\frac{\sqrt{n}}{\sqrt{2\pi} \cdot \sigma}\right) + \lim_{n \rightarrow \infty} \frac{1}{n} \log\left(\int_x^{\infty} e^{-n \cdot I(y)} dy\right)$$

$= 0$ LAPLACE'S PRINCIPLE HW 1.2 $\Rightarrow -\inf_{y \geq x} I(y) = I(x)$

THUS INDEED: $P\left(\frac{S_m}{n} \geq x\right) \approx e^{-n \cdot I(x)}$

SO THE UPPER BOUND ☆ WAS SHARP IN THIS SENSE (SEE PAGE 2 FOR NOTATION)

$Z(\lambda) := \mathbb{E}(e^{\lambda \cdot X})$, ASSUME THAT $\exists \underline{\lambda} < 0, \bar{\lambda} > 0$

SUCH THAT $Z(\lambda) < +\infty$ IF $\lambda \in [\underline{\lambda}, \bar{\lambda}]$ ☐

LEMMA: $\lambda \mapsto Z(\lambda)$ IS ANALYTIC ☐ ON $(\underline{\lambda}, \bar{\lambda})$,

moreover $\frac{d^m}{d\lambda^m} Z(\lambda) \stackrel{A}{=} \mathbb{E}(X^m \cdot e^{\lambda \cdot X})$

PROOF: $\frac{d^m}{d\lambda^m} e^{\lambda \cdot X} = X^m \cdot e^{\lambda \cdot X}$, LET $\lambda_0 \in (\underline{\lambda}, \bar{\lambda})$

THUS $\exists \varepsilon > 0 : [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] \subseteq (\underline{\lambda}, \bar{\lambda})$

$$e^{(\lambda_0 + M) \cdot X} = e^{\lambda_0 \cdot X} \cdot e^{M \cdot X} = \sum_{r=0}^{\infty} e^{\lambda_0 \cdot X} \cdot \frac{M^r \cdot X^r}{r!}$$

IF
 $|M| \leq \varepsilon$

NOTE: $\sum_{r=0}^{\infty} \frac{|M|^r \cdot |X|^r}{r!} = e^{|M| \cdot |X|} \leq e^{\varepsilon \cdot |X|} \leq e^{-\varepsilon \cdot X} + e^{\varepsilon \cdot X}$

THUS $\forall m \geq 0 : \sum_{r=0}^m \left| e^{\lambda_0 \cdot X} \cdot \frac{M^r \cdot X^r}{r!} \right| \stackrel{U}{\leq} \underbrace{e^{(\lambda_0 - \varepsilon) \cdot X} + e^{(\lambda_0 + \varepsilon) \cdot X}}_{< +\infty}$ ☐

THUS $\mathbb{E} \left(\sum_{r=0}^m \left| e^{\lambda_0 \cdot X} \cdot \frac{M^r \cdot X^r}{r!} \right| \right) \stackrel{V}{\leq} \mathbb{E} \left(\sum_{r=0}^m \left| e^{\lambda_0 \cdot X} \cdot \frac{M^r \cdot X^r}{r!} \right| \right) < +\infty$ ☐

THUS BY DOMINATED CONVERGENCE THM: ☐

$$\mathbb{E}(e^{(\lambda_0 + M) \cdot X}) = \mathbb{E} \left(\lim_{n \rightarrow \infty} \sum_{r=0}^n e^{\lambda_0 \cdot X} \cdot \frac{M^r \cdot X^r}{r!} \right) = \text{☐ ☐}$$

$$= \lim_{n \rightarrow \infty} \mathbb{E} \left(\sum_{r=0}^n e^{\lambda_0 \cdot X} \cdot \frac{M^r \cdot X^r}{r!} \right) = \lim_{n \rightarrow \infty} \sum_{r=0}^n \frac{M^r}{r!} \mathbb{E}(X^r \cdot e^{\lambda_0 \cdot X})$$

$$= \sum_{r=0}^{\infty} \frac{M^r}{r!} \mathbb{E}(X^r \cdot e^{\lambda_0 \cdot X}) \quad \text{IF } |M| \leq \varepsilon$$

$$\hat{I}(\lambda) = \ln(z(\lambda))$$

WANT: $\lambda \mapsto \hat{I}(\lambda)$ IS STRICTLY CONVEX

$$\hat{I}'(\lambda) \stackrel{\square}{=} \frac{z'(\lambda)}{z(\lambda)} = \frac{\mathbb{E}(X_1 \cdot e^{\lambda X_1})}{z(\lambda)}$$

$$\hat{I}''(\lambda) = \frac{z''(\lambda) \cdot z(\lambda) - (z'(\lambda))^2}{(z(\lambda))^2} = \frac{\mathbb{E}(X_1^2 \cdot e^{\lambda X_1})}{z(\lambda)} - (\hat{I}'(\lambda))^2$$

WHY DO WE HAVE $\hat{I}''(\lambda) > 0$?

NOTE: $z(0) = 1$, $\hat{I}(0) = 0$, $\hat{I}'(0) = \mathbb{E}(X_1)$

$$\hat{I}''(0) = \mathbb{E}(X_1^2) - (\mathbb{E}(X_1))^2 = \text{Var}(X_1) > 0 \quad \square$$

AND FOR $\lambda \neq 0$, $\hat{I}''(\lambda) = \text{Var}(\text{?}) \quad \square$

DEF: (MEASURE CHANGE, EXPONENTIALLY TILTED R.V.'S)

GIVEN X_1 AND $\lambda \in \mathbb{R}$ SUCH THAT $z(\lambda) < +\infty$,

DEFINE $X_1^{(\lambda)}$ TO BE A R.V. THAT SATISFIES

$$\mathbb{P}(X_1^{(\lambda)} \in A) \stackrel{\square}{=} \mathbb{E} \left(\frac{e^{\lambda X_1} \cdot \mathbb{1}[X_1 \in A]}{z(\lambda)} \right) \quad \square \quad \text{FOR ANY } A \subseteq \mathbb{R}$$

IN PARTICULAR:

$$F^{(\lambda)}(x) \stackrel{\square}{=} \mathbb{P}(X_1^{(\lambda)} \leq x) \stackrel{\text{B}}{=} \mathbb{E} \left(\frac{e^{\lambda X_1} \cdot \mathbb{1}[X_1 \leq x]}{z(\lambda)} \right)$$

NOTE: $F^{(\lambda)}$ IS INDEED A PROB. DISTRIBUTION F'N:
NON-DECREASING, RIGHT-CONTINUOUS,

$\lim_{x \rightarrow -\infty} F^{(\lambda)}(x) = 0$ MONOTONE CONVERGENCE THM

$\lim_{x \rightarrow +\infty} F^{(\lambda)}(x) = \mathbb{E} \left(\frac{e^{\lambda \cdot X'} \cdot \mathbb{1}[X' < +\infty]}{Z(\lambda)} \right) = \frac{Z'(\lambda)}{Z(\lambda)} = 1$

NOTE: $\mathbb{E} (g(X'^{(\lambda)})) \stackrel{C}{=} \mathbb{E} \left(g(X') \cdot \frac{e^{\lambda X'}}{Z(\lambda)} \right)$

NOTE: IF X' IS INTEGER-VALUED WITH

$P(X' = r) = P_r$
 $r \in \mathbb{Z}$

THEN

$P(X'^{(\lambda)} = r) \stackrel{D}{=} \frac{e^{\lambda r} \cdot P_r}{Z(\lambda)}$

NOTE: IF X' IS ABSOLUTELY CONTINUOUS WITH DENSITY FUNCTION $f(x)$ THEN

$P(X'^{(\lambda)} \leq x) \stackrel{E}{=} \int_{-\infty}^x \frac{e^{\lambda y}}{Z(\lambda)} \cdot f(y) dy$, THUS $X'^{(\lambda)}$ IS

ALSO ABS. CONTINUOUS WITH DENSITY

FUNCTION $f^{(\lambda)}(x) \stackrel{H}{=} \frac{e^{\lambda x}}{Z(\lambda)} \cdot f(x)$

NOW $Z'(\lambda) \stackrel{F}{=} \mathbb{E}(X' \cdot e^{\lambda X'})$, $Z''(\lambda) \stackrel{G}{=} \mathbb{E}(X'^2 \cdot e^{\lambda X'})$,

$\hat{I}'(\lambda) = \frac{Z'(\lambda)}{Z(\lambda)} = \mathbb{E} \left(X' \cdot \frac{e^{\lambda X'}}{Z(\lambda)} \right) \stackrel{I}{=} \mathbb{E}(X'^{(\lambda)})$

THUS:

$$\hat{I}''(\lambda) \stackrel{E}{=} \frac{z''(\lambda)}{z(\lambda)} - \left(\frac{z'(\lambda)}{z(\lambda)}\right)^2 = E((X^{(\lambda)})^2) - (E(X^{(\lambda)}))^2$$

$$\hat{I}''(\lambda) \stackrel{W}{=} \text{Var}(X^{(\lambda)}) > 0$$

INDEED:

\hat{I} STRICTLY CONVEX!

THUS BY PAGE 12-13:

$$\text{IF WE DEFINE } I(x) = \sup_{\lambda} \{ \lambda \cdot x - \hat{I}(\lambda) \}$$

$$\text{THEN INDEED } \hat{I}(\lambda) = \sup_x \{ \lambda \cdot x - I(x) \},$$

MOREOVER

$\lambda \mapsto \hat{I}'(\lambda)$ IS STRICTLY INCREASING, CONTINUOUS

$x \mapsto I'(x)$ IS ——— ——— ——— ——— ———

$$I' \stackrel{F}{=} (\hat{I}')^{-1}$$

AND IF WE DEFINE

$$\lambda^*(x) \stackrel{G}{=} (\hat{I}')^{-1}(x)$$

$$\text{THEN } E(X^{(\lambda^*(x))}) = \hat{I}'(\lambda^*(x)) = x \text{ AND}$$

$$I(x) = x \cdot \lambda^*(x) - \hat{I}(\lambda^*(x))$$

$$I''(x) = \frac{1}{\text{Var}(X^{(\lambda^*(x))})}$$

NOTE: $\hat{I}(0) = 0$, $\hat{I}'(0) = E(X) = m$, THUS

$$\lambda^*(m) = 0, \text{ THUS } I(m) = m \cdot 0 - \hat{I}(0) = 0$$

$$(I'(m))$$