

THUS:

$$\hat{I}''(\lambda) = \frac{z''(\lambda)}{z(\lambda)} - \left(\frac{z'(\lambda)}{z(\lambda)}\right)^2 = \mathbb{E}((X^{(\lambda)})^2) - (\mathbb{E}(X^{(\lambda)}))^2$$

$$\hat{I}''(\lambda) = \text{Var}(X^{(\lambda)}) > 0$$

INDEED:

\hat{I} STRICTLY CONVEX!

THUS BY PAGE 12-13:

IF WE DEFINE $I(x) = \sup_{\lambda} \{ \lambda \cdot x - \hat{I}(\lambda) \}$

THEN INDEED $\hat{I}(\lambda) = \sup_x \{ \lambda \cdot x - I(x) \}$,

MOREOVER

$\lambda \mapsto \hat{I}'(\lambda)$ IS STRICTLY INCREASING, CONTINUOUS

$x \mapsto I'(x)$ IS " " " " " "

$$I' = (\hat{I}')^{-1}$$

AND IF WE DEFINE

$$\lambda^*(x) := (\hat{I}')^{-1}(x)$$

THEN $\mathbb{E}(X^{(\lambda^*(x))}) = \hat{I}'(\lambda^*(x)) = x$ AND

$$I(x) = x \cdot \lambda^*(x) - \hat{I}(\lambda^*(x))$$

$$I''(x) = \frac{1}{\text{Var}(X^{(\lambda^*(x))})}$$

NOTE: $\hat{I}(0) = 0$, $\hat{I}'(0) = \mathbb{E}(X) = m$, THUS

$\lambda^*(m) = 0$, THUS $I(m) = m \cdot 0 - \hat{I}(0) = 0$
($I'(m)$)

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NOTE: IF $X \geq m$ THEN $\lambda^*(x) \geq \lambda^*(m) = 0$

THUS $I(x) = \max_{\lambda \geq 0} \{ \lambda \cdot x - \hat{I}(\lambda) \}$, THUS

BY PAGE 8: $P\left(\frac{S_m}{m} \geq x\right) \leq e^{-m \cdot I(x)}$

LEMMA: IF X AND Y ARE INDEPENDENT
AND $X^{(\lambda)}$ AND $Y^{(\lambda)}$ ARE — " —

THEN $(X + Y)^{(\lambda)} \sim X^{(\lambda)} + Y^{(\lambda)}$ 

PROOF: WE ONLY KNOW IT IN THE CASE WHEN
BOTH X AND Y ARE ABS. CONTINUOUS:

f IS THE DENSITY F'N OF X

g IS — " — — " — — " — OF Y

$f * g$ IS — " — — " — — " — OF $X + Y$

WANT: 

$$(f * g)^{(\lambda)} = f^{(\lambda)} * g^{(\lambda)}$$

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y) \cdot g(x-y) dy \quad (\text{CONVOLUTION})$$

NOTE: $Z_{X+Y}(\lambda) = \boxed{\text{HW 1. } Z(f)} = Z_X(\lambda) \cdot Z_Y(\lambda)$ 

$$(f * g)^{(\lambda)}(x) \stackrel{\text{HW 1.}}{=} \frac{e^{\lambda x}}{Z_{X+Y}(\lambda)} \cdot \int_{-\infty}^{\infty} f(y) \cdot g(x-y) dy =$$

$$= \int_{-\infty}^{\infty} \left(\frac{e^{\lambda y}}{Z_X(\lambda)} \cdot f(y) \right) \cdot \left(\frac{e^{\lambda(x-y)}}{Z_Y(\lambda)} \cdot g(x-y) \right) dy \stackrel{\text{HW 1.}}{=} (f^{(\lambda)} * g^{(\lambda)})(x)$$

REMINDER: X_1, X_2, \dots i.i.d., $S_n = X_1 + \dots + X_n$

$$Z(\lambda) = \mathbb{E}(e^{\lambda X_i}), \quad \hat{I}(\lambda) = \ln(Z(\lambda))$$

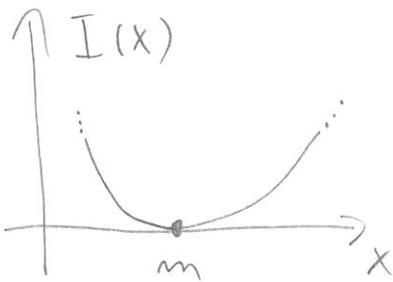
$$I(x) = \max_{\lambda} \{x\lambda - \hat{I}(\lambda)\} = x \cdot \lambda^*(x) - \hat{I}(\lambda^*(x)),$$

$$\lambda^*(x) = (\hat{I}')^{-1}(x), \quad \mathbb{E}(X_i^{\lambda^*(x)}) = x, \quad \text{WHERE}$$

$$\mathbb{P}(X_i^{\lambda^*(x)} \in A) = \mathbb{E}\left(\frac{e^{\lambda^*(x) X_i}}{Z(\lambda^*(x))} \cdot \mathbb{1}[X_i \in A]\right), \quad A \subseteq \mathbb{R}$$

KNOW: $x \mapsto I(x)$ IS STRICTLY CONVEX

$$m = \mathbb{E}(X_i) \quad I'(m) = 0, \quad I(m) = 0, \quad I''(x) > 0$$



THUS $I(x) > 0$ IF $x \neq m$

THM (H. CRAMÉR): $a < b \in \mathbb{R}$:

$$\textcircled{U} \limsup_{n \rightarrow \infty} \frac{1}{n} \ln \left(\mathbb{P}\left(\frac{S_n}{n} \in [a, b]\right) \right) \leq - \inf_{a \leq x \leq b} I(x)$$

$$\textcircled{L} \liminf_{n \rightarrow \infty} \frac{1}{n} \ln \left(\mathbb{P}\left(\frac{S_n}{n} \in (a, b)\right) \right) \geq - \inf_{a < x < b} I(x)$$

REMARK: IF $X_{Y_i} \sim \text{BER}(p)$ THEN

$$I(x) = \begin{cases} x \cdot \ln\left(\frac{x}{p}\right) + (1-x) \cdot \ln\left(\frac{1-x}{1-p}\right) & \text{IF } 0 \leq x \leq 1 \\ +\infty & \text{IF } x \notin [0, 1] \end{cases}$$

$$\inf_{1 \leq x \leq 2} I(x) = I(1) = \ln(p) \iff \mathbb{P}\left(\frac{S_m}{n} \geq 1\right) = p^n = e^{-n \cdot I(1)}$$

$$\inf_{1 \leq x \leq 2} I(x) = +\infty \iff \mathbb{P}\left(\frac{S_m}{n} > 1\right) = 0 = e^{-n \cdot \infty}$$

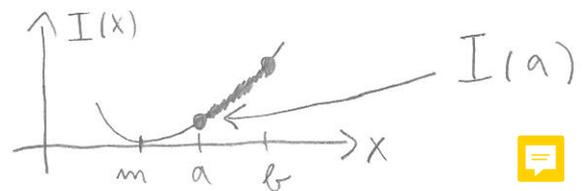
PROOF: (U): IF $m \in [a, b]$ THEN $\inf_{a \leq x \leq b} I(x) = I(m) = 0$

$$\text{THUS } \mathbb{P}\left(\frac{S_m}{n} \in [a, b]\right) \leq 1 = e^{-n \cdot I(m)} \quad \checkmark$$

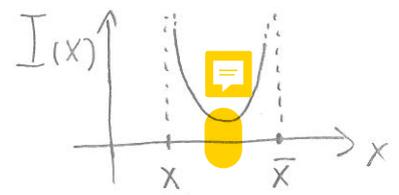
WITHOUT LOSS OF GENERALITY, ASSUME $m < a$:

$$\mathbb{P}\left(\frac{S_m}{n} \in [a, b]\right) \leq \mathbb{P}\left(\frac{S_m}{n} \geq a\right) \leq e^{-n \cdot I(a)} \quad (\text{PAGE 20})$$

$$\text{AND } I(a) = \inf_{a \leq x \leq b} I(x) \quad \checkmark$$



(L): IF $\underline{x} = \inf \{x : I(x) < +\infty\}$
 $\bar{x} = \sup \{x : I(x) < +\infty\}$



$$\text{THEN } x \in (\underline{x}, \bar{x}) \implies \boxed{I(x) < +\infty} \quad \boxed{|I'(x)| < +\infty}$$

(SINCE I IS CONVEX, SMOOTH)

IF $(a, b) \cap (\underline{x}, \bar{x}) = \emptyset$ THEN $\inf_{a < x < b} I(x) = +\infty$

THUS (L) TRIVIAALLY FOLLOWS.

WE WILL SHOW THAT IF $x \in (a, b) \cap (\underline{x}, \bar{x})$,

THEN $\liminf_{n \rightarrow \infty} \frac{1}{n} \ln (P(\frac{S_n}{n} \in (a, b))) \geq -I(x)$



AS SOON AS WE SHOW (★), THE PROOF OF (L) FOLLOWS.

NOTE: IF ϵ IS SMALL THEN $[x-\epsilon, x+\epsilon] \subseteq (a, b)$

LET $\lambda^* = \lambda^*(x) := I'(x)$, THEN $E(X_i^{(\lambda^*)}) = x$

ALSO NOTE: $X_1^{(\lambda^*)} + \dots + X_n^{(\lambda^*)} \stackrel{\text{□}}{\sim} S_n^{(\lambda^*)}$ (SEE PAGE 20)

WEAK LAW OF LARGE NUMBERS

THUS $\lim_{n \rightarrow \infty} P(\frac{S_n^{(\lambda^*)}}{n} \in [x-\epsilon, x+\epsilon]) \stackrel{\text{□}}{=} 1$, BUT

$P(S_n^{(\lambda^*)} \in [n \cdot (x-\epsilon), n \cdot (x+\epsilon)]) =$ (SEE PAGE 17)

$E\left(\frac{e^{\lambda^* \cdot S_n}}{Z(\lambda^*)^n} \cdot \mathbb{I}[S_n \in [n \cdot (x-\epsilon), n \cdot (x+\epsilon)]]\right) \leq$

$\frac{e^{\lambda^* \cdot (x+\epsilon) \cdot n}}{(Z(\lambda^*))^n} \cdot P\left(\frac{S_n}{n} \in [x-\epsilon, x+\epsilon]\right) \leq P\left(\frac{S_n}{n} \in (a, b)\right)$

$\lambda^* > 0$
SINCE
 $m < a < x$

THUS: $\forall \epsilon > 0$ (SMALL):

$$0 = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left(\mathbb{P} \left(\frac{S_n^{(\lambda^*)}}{n} \in [x - \epsilon, x + \epsilon] \right) \right) \leq$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln \left(\frac{e^{\lambda^* \cdot (x + \epsilon) \cdot n}}{(Z(\lambda^*))^n} \right) + \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left(\mathbb{P} \left(\frac{S_n}{n} \in (a, b) \right) \right)$$

SEE PAGE 21

$$= \lambda^* \cdot x + \lambda^* \cdot \epsilon - \hat{I}(\lambda^*) = I(x) + \lambda^* \cdot \epsilon$$

SINCE IT HOLDS FOR ANY $\epsilon > 0$, WE ARE DONE WITH THE PROOF OF \star . WE USED MEASURE CHANGE.

REMARK: IF $x \in (\underline{x}, \bar{x})$ THEN

$$\lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left(\mathbb{P} \left(\frac{S_n}{n} \in [x - \epsilon, x + \epsilon] \right) \right) = -I(x)$$

HEURISTICALLY: $\mathbb{P} \left(\frac{S_n}{n} \approx x \right) \approx e^{-n \cdot I(x)}$

$I(x)$ IS THE "COST" OF $\frac{S_n}{n} \approx x$

CRAMÉR: IF THE UNLIKELY EVENT

$\frac{S_n}{n} \in [a, b]$ OCCURS, THEN IT WILL

OCCUR USING THE LEAST COSTLY

STRATEGY.

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THE BEST STRATEGY TO ACHIEVE

$\frac{S_m}{n} \approx x$ IS TO TILT WITH $\lambda^*(x)$: 

CLAIM: IF X_i IS INTEGER-VALUED

WITH $P(X_i = r) = P_r$, $X > m$, THEN 

$$\lim_{n \rightarrow \infty} P(X_{i,1} = r \mid S_m \geq n \cdot x) = P_r = \frac{e^{\lambda^*(x) \cdot r} \cdot P_r}{Z(\lambda^*(x))}$$


PROOF (NON-RIGOROUS): FIRST NOTE THAT IF $\lambda^* = \lambda^*(x)$

$$P(X_{i,1}^{(\lambda^*)} = r \mid S_m^{(\lambda^*)} = m) = P(X_{i,1} = r \mid S_m = m)$$

 HW 2.3

THUS $P(X_{i,1} = r \mid S_m \geq n \cdot x) \approx$  CRAMÉR

$$P(X_{i,1} = r \mid S_m \approx n \cdot x) \approx$$


$$P(X_{i,1}^{(\lambda^*)} = r \mid S_m^{(\lambda^*)} \approx n \cdot x) =$$


$$= \frac{P_r^{(\lambda^*)} \cdot P(X_2^{(\lambda^*)} + \dots + X_m^{(\lambda^*)} \approx n \cdot x - r)}{P(X_1^{(\lambda^*)} + \dots + X_m^{(\lambda^*)} \approx n \cdot x)}$$




WEAK LAW OF LARGE NUMBERS \approx

$$\frac{P_r^{(\lambda^*)} \cdot 1}{1} = P_r^{(\lambda^*)}$$