EX: $\boxed{BIN\left(n, \frac{\lambda}{n}\right) \implies POI(\lambda)}$ AS $n \to \infty$

PROOF. LET $X_n \sim BIN\left(n, \frac{\lambda}{n}\right)$, $X \sim POI(\lambda)$

$$\lim_{n \to \infty} \underbrace{\binom{n}{k} \cdot \left(\frac{\lambda}{n}\right)^k \cdot \left(1 - \frac{\lambda}{n}\right)^{n-k}}_{\mathbb{P}(X_n = k)} = \underbrace{e^{-\lambda} \cdot \frac{\lambda^k}{k!}}_{\mathbb{P}(X = k)}, \quad k \in \mathbb{N}$$

THUS BY CLAIM STATED ON PAGE 44,

WE HAVE $X_n \implies X$ ✓

CENTRAL LIMIT THEOREM (C.L.T.):

$X_1, X_2, \dots$ I.I.D. $\mathbb{E}(X_i) = m$, $Var(X_i) = \sigma^2 < +\infty$

$S_n = X_1 + \dots + X_n$, THEN

$$\boxed{\frac{S_n - \mathbb{E}(S_n)}{\sqrt{Var(S_n)}} \implies \mathcal{N}(0,1)}$$

PROOF: IN A FEW WEEKS

$$\boxed{\psi(x) := \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}}$$

THAT IS: $\forall x \in \mathbb{R}$:

$$\lim_{n \to \infty} \mathbb{P}\left(\frac{S_n - n \cdot m}{\sigma \cdot \sqrt{n}} \leq x\right) = \Phi(x) = \int_{-\infty}^{x} \psi(y)\,dy$$

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WE WILL NOW PROVE A SPECIAL CASE:
WHEN $\boxed{X_i \sim EXP(1)}$. AT THE SAME
TIME, WE WILL PROVE STIRLING'S FORMULA:

$$\boxed{\lim_{n \to \infty} \frac{\sqrt{2\pi} \cdot n^{n+\frac{1}{2}} \cdot e^{-n}}{n!} = 1}$$

NOTE: $\mathbb{E}(X_i) = 1$, $Var(X_i) = 1$

$$\mathbb{E}(S_n) = n, \quad Var(S_n) = n$$

WANT: $\boxed{\dfrac{S_n - n}{\sqrt{n}} \Rightarrow \mathcal{N}(0,1)}$ $\boxed{HW\ 3.3\ (a)}$

KNOW: P.D.F. OF S_n : $f_n(x) \underset{A}{=} e^{-x} \cdot \dfrac{x^{n-1}}{(n-1)!} \cdot \mathbb{1}[x \geq 0]$

THUS: P.D.F. OF $(S_n - n)/\sqrt{n}$ IS :

$$g_n(x) \underset{D}{=} \sqrt{n} \cdot f_n(n + \sqrt{n} \cdot x) =$$

$$= \sqrt{n} \cdot e^{-(n + \sqrt{n} \cdot x)} \cdot \frac{(n + \sqrt{n} \cdot x)^{n-1}}{(n-1)!} \cdot \mathbb{1}[n + \sqrt{n} \cdot x \geq 0]$$

$$\underset{B}{=} \frac{n^{3/2} \cdot e^{-(n + \sqrt{n} \cdot x)} \cdot (n + \sqrt{n} \cdot x)^{n-1}}{n!} \cdot \mathbb{1}[x \geq -\sqrt{n}]$$

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DEF: LET $\boxed{\tilde{g}_n(x) := g_n(x) \cdot \dfrac{n!}{\sqrt{2\pi} \cdot n^{n+1/2} \cdot e^{-n}}}$

LEMMA: $\forall\, k \in \mathbb{R}+$ $\boxed{\tilde{g}_n(x) \longrightarrow \dfrac{1}{\sqrt{2\pi}} e^{-x^2/2}}$

AS $n \to \infty$, UNIFORMLY FOR ALL $x \in [-k, k]$

PROOF: $\tilde{g}_n(x) = \dfrac{n^{3/2} \cdot e^{-(n+\sqrt{n}\cdot x)} \cdot (n + \sqrt{n}\cdot x)^{n-1}}{\sqrt{2\pi} \cdot n^{n+1/2} \cdot e^{-n}} =$

$= \dfrac{1}{\sqrt{2\pi}} \cdot \dfrac{e^{-(n+\sqrt{n}\cdot x)}}{e^{-n}} \cdot \dfrac{(n + \sqrt{n}\cdot x)^{n-1}}{n^{n-1}} =$

$= \dfrac{1}{\sqrt{2\pi}} \cdot e^{-\sqrt{n}\cdot x} \cdot \left(1 + \dfrac{x}{\sqrt{n}}\right)^{n-1} =$

$= \dfrac{\left(1 + \dfrac{x}{\sqrt{n}}\right)^{-1}}{\sqrt{2\pi}} \cdot \left(e^{-x/\sqrt{n}} \cdot \left(1 + \dfrac{x}{\sqrt{n}}\right)\right)^{n}$

IT REMAINS TO SHOW:

$\ln\left(\lim_{n\to\infty}\left(e^{-x/\sqrt{n}} \cdot \left(1 + \dfrac{x}{\sqrt{n}}\right)\right)^{n}\right) = \lim_{n\to\infty} n \cdot \left(-\dfrac{x}{\sqrt{n}} + \ln\left(1 + \dfrac{x}{\sqrt{n}}\right)\right)$

$= \lim_{n\to\infty} n \cdot \left(-\dfrac{x}{\sqrt{n}} + \dfrac{x}{\sqrt{n}} - \dfrac{1}{2}\dfrac{x^2}{n}\right) = -\dfrac{1}{2}x^2 \quad \checkmark$

$\boxed{\ln(1+y) = y - \dfrac{1}{2}y^2 + \mathcal{O}(y^3)}$

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LET $\quad \alpha_m := \dfrac{m!}{\sqrt{2\pi} \cdot m^{m+1/2} \cdot e^{-m}}$

NOTE:

$\tilde{g}_m(x) = g_m(x) \cdot \alpha_m$

THM (DE MOIVRE, STIRLING): $\quad \lim\limits_{m \to \infty} \alpha_m = 1$

PROOF: ENOUGH TO SHOW THAT $\forall \varepsilon > 0$:

$$1 - \varepsilon \leq \liminf_{m \to \infty} \alpha_m \leq \limsup_{m \to \infty} \alpha_m \leq \frac{1}{1-\varepsilon}$$

LET $K := \dfrac{1}{\sqrt{\varepsilon}}$ AND NOTE: [CHEBYSHEV]

$$\int_{-K}^{K} g_m(x)\, dx = \mathbb{P}\left(\left| \frac{S_m - \mathbb{E}(S_m)}{\sqrt{\mathrm{Var}(S_m)}} \right| \leq K \right) \geq 1 - \frac{1}{K^2} = 1 - \varepsilon$$

$$\int_{-K}^{K} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx \geq 1 - \varepsilon \quad\text{(CHEBYSHEV)} \qquad , \quad \text{THUS}$$

$$\alpha_m \cdot (1-\varepsilon) \leq \alpha_m \cdot \int_{-K}^{K} g_m(x)\, dx = \int_{-K}^{K} \tilde{g}_m(x)\, dx \longrightarrow \int_{-K}^{K} \varphi(x)\, dx \leq 1$$

THUS $\quad \limsup\limits_{m \to \infty} \alpha_m \leq \dfrac{1}{1-\varepsilon} \quad,\quad$ AND

$$\alpha_m = \alpha_m \cdot \int_{-\infty}^{\infty} g_m(x)\, dx = \int_{-\infty}^{\infty} \tilde{g}_m(x)\, dx \geq \int_{-K}^{K} \tilde{g}_m(x)\, dx \longrightarrow \underbrace{\int_{-K}^{K} \varphi(x)\, dx}_{\geq 1-\varepsilon}$$

THUS $\quad \liminf\limits_{m \to \infty} \alpha_m \geq 1 - \varepsilon$ ✓

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THUS $\boxed{g_n(x) \to \varphi(x)}$ (POINTWISE) $\forall x \in \mathbb{R}$.

IN ORDER TO COMPLETE THE PROOF OF C.L.T. FOR EXP(1) SUMMANDS, WE NEED:

LEMMA (SCHEFFÉ): IF f_n IS A P.D.F. FOR EACH n AND f IS A P.D.F. AND IF

$$\boxed{\lim_{n \to \infty} f_n(x) = f(x)}$$ (POINTWISE) $\forall x \in \mathbb{R}$, THEN

(A) $\lim_{n \to \infty} \int_{-\infty}^{\infty} |f_n(x) - f(x)| \, dx = 0 \quad \left(f_n \xrightarrow{L_1} f \right)$

(B) $\boxed{F_n \Rightarrow F}$ $\quad F_n(x) = \int_{-\infty}^{x} f_n(y) \, dy$, $\quad F(x) = \int_{-\infty}^{x} f(y) \, dy$

PROOF: (A) IDEA: USE $\boxed{|f_n - f| = f_n + f - 2 \cdot (f \wedge f_n)}$ AND DOMINATED CONVERGENCE.

$0 \leq f(x) \wedge f_n(x) \leq f(x) \quad$ AND $\int_{-\infty}^{\infty} f(x) \, dx = 1$

MOREOVER $\lim_{n \to \infty} f(x) \wedge f_n(x) = f(x)$, THUS

$$\boxed{\lim_{n \to \infty} \int_{-\infty}^{\infty} (f_n(x) \wedge f(x)) \, dx = 1}$$ BY DOMINATED CONVERGENCE.

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$$\int_{-\infty}^{\infty} |f_m(x) - f(x)| \, dx = \underbrace{\int_{-\infty}^{\infty} f_m(x) \, dx}_{1} + \underbrace{\int_{-\infty}^{\infty} f(x) \, dx}_{1} - 2 \cdot \underbrace{\int_{-\infty}^{\infty} (f_m(x) \wedge f(x)) \, dx}_{m \to \infty \quad \rightarrow 1}$$

THUS $\lim\limits_{m \to \infty} \int_{-\infty}^{\infty} |f_m(x) - f(x)| \, dx = 2 - 2 = 0$ ✓

Ⓑ $\left| F_m(x) - F(x) \right| = \left| \int_{-\infty}^{x} (f_m(x) - f(x)) \, dx \right| \leq$

$\int_{-\infty}^{x} |f_m(x) - f(x)| \, dx \leq \int_{-\infty}^{\infty} |f_m(x) - f(x)| \, dx \xrightarrow[m \to \infty]{} 0$

THM (CRAMÉR-SLUTSKY)

IF $\boxed{X_m \Rightarrow X}$ AND $\boxed{Y_m \Rightarrow C}$ ← DETERMINISTIC CONSTANT THEN

Ⓐ $X_m + Y_m \Rightarrow X + C$

Ⓑ $X_m \cdot Y_m \Rightarrow X \cdot C$

Ⓒ $X_m / Y_m \Rightarrow X / C$ IF $C \neq 0$.

PROOF: WE WILL ONLY PROVE Ⓐ

SEE NEXT PAGE.

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(A) LET $F_n(x) = \mathbb{P}(X_{,n} + Y_n \leq x)$

$$F(x) = \mathbb{P}(X + C \leq x)$$

ENOUGH TO SHOW THAT $\forall x \in \mathbb{R}, \forall \varepsilon > 0$

$$\boxed{F(x-\varepsilon) \leq \liminf_{n \to \infty} F_n(x) \leq \limsup_{n \to \infty} F_n(x) \leq F(x+\varepsilon)}$$

BECAUSE IF $F(x_-) = F(x)$ THEN THIS

WILL IMPLY $\boxed{\lim_{n \to \infty} F_n(x) = F(x)}$

GIVEN x, ε : LET $0 < \delta \leq \varepsilon$: $\boxed{\mathbb{P}(X + C = x - \delta) = 0}$

$$\mathbb{P}(X_{,n} + Y_n \leq x) = \mathbb{P}((X_{,n} + C) + (Y_n - C) \leq x) \geq$$

$$\geq \mathbb{P}((X_{,n} + C) \leq x - \delta, \; Y_n - C \leq \delta) \geq$$

$$\underbrace{\mathbb{P}(X_{,n} + C \leq x - \delta)}_{n \nearrow^{\infty}} - \underbrace{\mathbb{P}(Y_n > C + \delta)}_{n \nearrow^{\infty} \to 0}$$

$$\longrightarrow \mathbb{P}(X + C \leq x - \delta) = F(x-\delta) \geq F(x-\varepsilon)$$

THIS SHOWS $\boxed{F(x-\varepsilon) \leq \liminf_{n \to \infty} F_n(x)}$

THE REST OF THE PROOF IS ANALOGOUS
  AND WE OMIT IT.

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