Midterm Exam 2 - May 20, 2026, Limit thms. of probab.

Family name ______________________ **Given name** ______________________

Signature ______________________ **Neptun Code** ______________________

No calculators or electronic devices are allowed. One formula sheet with 15 formulas is allowed.

1. Let $p \in (0, \frac{1}{2})$. Let $X_1, X_2, \ldots$ denote i.i.d. random variables with distribution

$$\mathbb{P}(X_i = +1) = p, \qquad \mathbb{P}(X_i = -1) = p, \qquad \mathbb{P}(X_i = 0) = 1 - 2p.$$

Let $W_0 = 0$ and $W_n = X_1 + \cdots + X_n$. Let $\mathcal{T}_0 := 0$ and let $\mathcal{T}_k := \min\{n > \mathcal{T}_{k-1} : W_n = 0\}$. For any $k \in \mathbb{N}_+$ let $\mathcal{T}_k^* := \min\{n : W_n = k\}$. Let $z \in \mathbb{C}$ with $|z| \leq 1$.

   (a) (3 points) Find $\mathbb{E}[z^{\mathcal{T}_1^*}]$. *Hint:* You will have to solve a quadratic equation.

   (b) (1 points) Find $\mathbb{E}[z^{\mathcal{T}_1}]$ by expressing it in terms of $\mathbb{E}[z^{\mathcal{T}_1^*}]$.

   (c) (1 point) Find $\mathbb{E}[z^{\mathcal{T}_k}]$.

   (d) (3 points) Find the value of $\beta \in \mathbb{R}_+$ such that

$$\mathcal{T}_k / k^\beta \Rightarrow \mathcal{T}$$

   as $k \to \infty$ (where $\mathcal{T}$ is a non-degenerate random variable) and find the characteristic function of $\mathcal{T}$.

2. (7 points) Let X_i denote i.i.d. random variables with p.d.f. $f(x) = 2x\mathbb{1}[0 < x < 1]$.
Let $S_n = \sum_{k=1}^n k^3 X_k$. Find α, b, β such that

$$\frac{S_n - bn^\beta}{n^\alpha}$$

weakly converges to a non-degenerate probability distribution and identify the limiting distribution.
Instruction: In case you use a theorem learnt in class, check the conditions of the theorem.
Hint: In your calculation you may use without proof that for any $\gamma > 0$ we have

$$1^\gamma + 2^\gamma + \cdots + n^\gamma = \frac{n^{\gamma+1}}{\gamma+1} + \mathcal{O}(n^\gamma)$$