Midterm Exam 2 – Solutions

May 20, 2026

Problem 1

Let

$$\mathbb{P}(X_i = 1) = p, \qquad \mathbb{P}(X_i = -1) = p, \qquad \mathbb{P}(X_i = 0) = 1 - 2p,$$

where $0 < p < 1/2$. Define

$$W_n = X_1 + \cdots + X_n, \qquad W_0 = 0.$$

Also define

$$T_0 := 0, \qquad T_k := \min\{n > T_{k-1} : W_n = 0\},$$

and

$$T_k^* := \min\{n : W_n = k\}.$$

Let

$$\varphi(z) := \mathbb{E}[z^{T_1^*}], \qquad |z| \le 1.$$

(a) Computing $\mathbb{E}[z^{T_1^*}]$

Condition on the first step of this lazy random walk.

- If $X_1 = 1$, then $T_1^* = 1$.
- If $X_1 = 0$, then after one step we are again at 0, hence the remaining time has the same distribution as T_1^*.
- If $X_1 = -1$, then to hit 1 we first have to go from -1 to 0, and then from 0 to 1. By the strong Markov property, these two times are independent and both distributed as T_1^*.

Therefore

$$\varphi(z) = pz + (1 - 2p)z\varphi(z) + pz\varphi(z)^2.$$

Hence φ satisfies the quadratic equation

$$pz\varphi(z)^2 + \big((1 - 2p)z - 1\big)\varphi(z) + pz = 0.$$

Solving for $\varphi(z)$ gives

$$\varphi(z) = \frac{1 - (1 - 2p)z \pm \sqrt{(1 - (1 - 2p)z)^2 - 4p^2z^2}}{2pz}.$$

Since $\varphi(0) = 0$, we must choose the minus sign. Thus

$$\boxed{\mathbb{E}[z^{T_1^*}] = \frac{1 - (1 - 2p)z - \sqrt{(1 - (1 - 2p)z)^2 - 4p^2z^2}}{2pz}.}$$

(b) Computing $\mathbb{E}[z^{T_1}]$

Again condition on the first step.

- If $X_1 = 0$, then $T_1 = 1$.
- If $X_1 = 1$ or $X_1 = -1$, then after the first step we need an additional time distributed as T_1^* to return to 0.

Therefore

$$\mathbb{E}[z^{T_1}] = (1-2p)z + 2pz\,\varphi(z).$$

Substituting the formula from part (a),

$$\mathbb{E}[z^{T_1}] = 1 - \sqrt{(1-(1-2p)z)^2 - 4p^2z^2}.$$

Since

$$(1-(1-2p)z)^2 - 4p^2z^2 = (1-z)(1-(1-4p)z),$$

we obtain

$$\boxed{\mathbb{E}[z^{T_1}] = 1 - \sqrt{(1-z)(1-(1-4p)z)}.}$$

(c) Computing $\mathbb{E}[z^{T_k}]$

The increments

$$T_1,\ T_2 - T_1,\ \dots,\ T_k - T_{k-1}$$

are i.i.d. by the strong Markov property, each having the same distribution as T_1. Hence

$$T_k = \sum_{j=1}^{k}(T_j - T_{j-1})$$

is a sum of k i.i.d. copies of T_1. Therefore

$$\boxed{\mathbb{E}[z^{T_k}] = \left(\mathbb{E}[z^{T_1}]\right)^k = \left(1 - \sqrt{(1-z)(1-(1-4p)z)}\right)^k.}$$

(d) Scaling limit of T_k

We study the characteristic function of T_k/k^β. For $t > 0$,

$$\mathbb{E}\left[e^{itT_k/k^\beta}\right] = \left(\mathbb{E}\left[e^{itT_1/k^\beta}\right]\right)^k.$$

Using part (b),

$$\mathbb{E}\left[e^{-T_1/k^\beta}\right] = 1 - \sqrt{\left(1-e^{it/k^\beta}\right)\left(1-(1-4p)e^{it/k^\beta}\right)}.$$

As $k \to \infty$,

$$1 - e^{it/k^\beta} \sim \frac{-it}{k^\beta},$$

and

$$1-(1-4p)e^{it/k^\beta} \to 4p.$$

Hence

$$\mathbb{E}\left[e^{itT_1/k^\beta}\right] = 1 - \frac{2\sqrt{-pit}}{k^{\beta/2}} + o\left(k^{-\beta/2}\right).$$

Therefore

$$\mathbb{E}\left[e^{itT_k/k^\beta}\right] = \left(1 - \frac{2\sqrt{-pit}}{k^{\beta/2}} + o(k^{-\beta/2})\right)^k.$$

To obtain a non-degenerate limit, we need

$$\boxed{\beta = 2.}$$

Then

$$\mathbb{E}\left[e^{itT_k/k^2}\right] \to e^{-2\sqrt{-pit}}.$$

Thus

$$\boxed{\frac{T_k}{k^2} \Rightarrow T,}$$

where

$$\boxed{\varphi_T(t) = \mathbb{E}[e^{itT}] = e^{-2\sqrt{p}\sqrt{-it}},}$$

thus T is a constant multiple of the standard Lévy distribution.

Problem 2

Let $X_1, X_2, \ldots$ be i.i.d. random variables with density

$$f(x) = 2x\mathbf{1}_{(0,1)}(x).$$

Define

$$S_n = \sum_{k=1}^n k^3 X_k.$$

We seek constants α, b, β such that

$$\frac{S_n - bn^\beta}{n^\alpha}$$

converges weakly to a non-degenerate distribution.

Step 1: Mean and variance

We compute

$$\mathbb{E}[X_1] = \int_0^1 2x^2\,dx = \frac{2}{3},$$

and

$$\mathbb{E}[X_1^2] = \int_0^1 2x^3\,dx = \frac{1}{2}.$$

Therefore

$$\mathrm{Var}(X_1) = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}.$$

Hence

$$\mathbb{E}[S_n] = \frac{2}{3}\sum_{k=1}^n k^3.$$

Using

$$\sum_{k=1}^n k^3 = \frac{n^4}{4} + O(n^3),$$

we obtain

$$\mathbb{E}[S_n] = \frac{n^4}{6} + O(n^3).$$

Thus the correct centering is

$$\boxed{b = \frac{1}{6}, \qquad \beta = 4.}$$

Next,

$$\operatorname{Var}(S_n) = \sum_{k=1}^{n} k^6 \operatorname{Var}(X_k) = \frac{1}{18} \sum_{k=1}^{n} k^6.$$

Using

$$\sum_{k=1}^{n} k^6 = \frac{n^7}{7} + O(n^6),$$

we get

$$\operatorname{Var}(S_n) = \frac{n^7}{126} + O(n^6).$$

Hence the fluctuations are of order $n^{7/2}$, so

$$\boxed{\alpha = \frac{7}{2}.}$$

Step 2: Applying Lindeberg's theorem

Define the triangular array

$$Y_{n,k} := \frac{k^3(X_k - 2/3)}{n^{7/2}}, \qquad 1 \le k \le n.$$

Then

$$\sum_{k=1}^{n} Y_{n,k} = \frac{S_n - (1/6)n^4}{n^{7/2}} + o(1).$$

We compute the variance:

$$\sum_{k=1}^{n} \operatorname{Var}(Y_{n,k}) = \frac{1}{n^7} \cdot \frac{1}{18} \sum_{k=1}^{n} k^6 \to \frac{1}{126}.$$

We now verify the Lindeberg condition. Since $0 \le X_k \le 1$,

$$|X_k - 2/3| \le 1,$$

so

$$|Y_{n,k}| \le \frac{k^3}{n^{7/2}} \le \frac{1}{\sqrt{n}}.$$

Hence for every fixed $\varepsilon > 0$,

$$|Y_{n,k}| < \varepsilon$$

for all sufficiently large n. Therefore

$$\sum_{k=1}^{n} \mathbb{E}\Big[Y_{n,k}^2 \mathbf{1}_{\{|Y_{n,k}| > \varepsilon\}}\Big] = 0$$

for all large enough n. Thus the Lindeberg condition holds.

By Lindeberg's Central Limit Theorem,

$$\sum_{k=1}^{n} Y_{n,k} \Rightarrow N\left(0, \frac{1}{126}\right).$$

Consequently,

$$\boxed{\frac{S_n - (1/6)n^4}{n^{7/2}} \Rightarrow N\left(0, \frac{1}{126}\right).}$$

Equivalently,

$$\frac{\sqrt{126}}{n^{7/2}}\left(S_n - \frac{n^4}{6}\right) \Rightarrow N(0,1).$$