July 11, 2014

Solutions of Math 302 HW assignment 1.

1. Five race horses enter a race (Seabiscuit, Red Rum, Desert Orchid, Black Caviar, Overdose). The horses are equally good, so all outcomes of the race are equally likely.

   (a) (2 mark) How many possible outcomes are there in which Desert Orchid is first and Black Caviar is last?

   Solution: The locations of these two horses are fixed, the remaining three horses can be freely permuted on the remaining three locations, thus the answer is 3!.

   (b) (2 mark) What is the probability that Red Rum is not the last one?

   Solution: The total number of possible outcomes of the race is 5! and the number of outcomes where Red Rum is the last one is 4!, thus the number of outcomes where Red Rum is not the last one is $5! - 4!$. The probability that Red Rum is not not the last one is $\frac{5!-4!}{5!} = 1 - \frac{1}{5}$.

   (c) (3 marks) How many possible outcomes are there in which Overdose beats Desert Orchid, who beats Red Rum?

   Solution: The relative order of these three horses can produce $3! = 6$ different possible outcomes, thus for each outcome π of the race in which O beats D, who beats R, we can generate altogether 6 outcomes of the race where the location of the rest of the horses is the same as in π. Thus we can partition the set of possible outcomes of the race into blocks of 6 elements, and each of these blocks contains exactly one outcome in which O beats D, who beats R. Thus the answer is $\frac{5!}{6}$.

2. I have 12 LEGO blocks: 3 green, 5 red, 4 blue. How many different LEGO towers can I build . . .

   (a) (2 marks) if I want the block on top to be red?

   Solution: Permutation with categories. The red on top is fixed, so I can play with 3 green, 4 red, 4 blue, altogether $3 + 4 + 4 = 11$ LEGO blocks. The answer is $\frac{11!}{3!4!4!}$.

   (b) (3 marks) if I want all the red blocks to be higher than all the blue blocks?

   Solution: The relative order of the blue and red blocks is completely determined, so the answer is the same as the number LEGO towers that I can build from 3 green and $5 + 4 = 9$ purple blocks. The answer is $\frac{12!}{3!9!}$.

   (c) (3 marks) if I don't want the three bottom blocks to be green?

   Solution: The height of the tower is 12. There are 12 locations, I have to choose a colour for each of the locations. First I choose the location of the green blocks. The three bottom blocks cannot be green, so I have 9 possible locations for my 3 green blocks. The number of ways to choose the locations of the green LEGO blocks is $\binom{9}{3}$. Given this choice, there are 9 possible remaining locations for my red and blue blocks, so after fixing the green choice, I have $\binom{9}{5}$ ways to choose the locations of red LEGO blocks (which already forces me to insert the remaining 4 blue blocks into the remaining 4 locations). The answer is $\binom{9}{3}\binom{9}{5}$.

3. Three families go to see a movie together. The Smiths have 4 family members, the Taylors have 5 and the Millers have 4. Each family has two parents and the rest are children (all distinguishable). One row in the cinema has 13 seats, so the three families together occupy a row. How many ways can they sit . . .

   (a) (3 marks) if each parent wants to sit next to their spouse?

   Solution: First let us "merge" each of the married couples into one unit. Now we have $13 - 3 = 10$ units, so the number of ways to arrange these units in a row is 10!. Now we split the married units again. The answer to the original question is $10! \cdot 2^3$ because each of the three married couples can actually be seated in two different ways even if they sit next to each other.

(b) (3 marks) if each of the families want to sit together?

Solution: Now we merge each of the three families into a unit. We have 3 units with 3! possible ways to arrange them in a row. Then we split the units again and permute the family members of each family while keeping them together. The answer to this question is 3!4!5!4!.

(c) (4 marks) if no two Millers are allowed to sit next to each other?

Solution: Let us first colour all the Millers green and all the others red. We have 4 green and 9 red people. Think about the green guys as walls. We have to put 3 red guys in between the green guys to separate them. We have 6 more indistinguishable red guys that we can freely arrange in the 5 slots created by the 4 walls. There are $\binom{6+4}{4}$ ways to arrange 4 walls and 6 indistinguishable guys. Therefore there are $\binom{10}{4}$ ways to seat 4 green and 9 red guys in such a way that we don't see two consecutive green guys. Now we wash away the red and green paint, i.e., we take into consideration that the 4 green and 9 red people are in fact distinguishable and can be permuted within their colour group. Thus the answer is $\binom{10}{4}4!9!$.

4. Google "list of poker hands wikipedia". What is the probability that a poker hand is...

(a) (3 marks) a "straight flush"?

Solution: A straight flush is a hand that contains five consecutive cards, all of the same suit. There are 4 ways to choose the suit. Aces can play high or low in straights and straight flushes! Thus we have $13-4$ straights plus one where the Ace plays the role of the lowest card rather than the highest, altogether 10. The answer to the question is $\frac{10\cdot 4}{\binom{52}{5}}$.

(b) (3 marks) a "flush" (but not a straight flush)?

Solution: A flush is a poker hand where all five cards are of the same suit, but not in sequence. There are 4 ways to choose the suit and $\binom{13}{5}$ ways to choose a hand of five cards from the set of cards with that suit. We also have to discard the 40 "straight flush" hands, so the answer is $\frac{4\binom{13}{5}-40}{\binom{52}{5}}$.

(c) (4 marks) a "two pairs" ?

Solution: We choose the suits of the two pairs: $\binom{13}{2}$. There are 11 suits remaining, we choose one for the fifth card. Given the suit of a pair, there are $\binom{4}{2}$ ways to choose a pair from with that suit. Given the suit of the fifth card, there are 4 ways to choose the card. The final answer is $\frac{\binom{13}{2}\cdot 11\cdot\binom{4}{2}^2\cdot 4}{\binom{52}{5}}$.

5. Use the binomial theorem to...

(a) (2 marks) find the closed form of $\sum_{j=0}^{50}\binom{50}{j}(-1)^j$.

Solution: $\sum_{j=0}^{50}\binom{50}{j}(-1)^j = \sum_{j=0}^{50}\binom{50}{j}(-1)^j\cdot 1^{50-j} = (-1+1)^{50} = 0$.

(b) (2 marks) expand $(x+2y)^{80}$ as a sum of monomials.

Solution: $(x+2y)^{80} = \sum_{k=0}^{80}\binom{80}{k}x^k(2y)^{80-k} = \sum_{k=0}^{80}2^{80-k}\binom{80}{k}\cdot x^k y^{80-k}$

6. If 18 orphan cats are divided among 6 cat orphanages, how many divisions are possible ...

(a) (2 mark) if the cats are distinguishable and no further restriction is imposed?

Solution: Each cat has 6 choices, so by the multiplication trick the answer is 6^{18}.

(b) (3 marks) if the cats are distinguishable and each orphanage must receive 3 cats?

Solution: $\binom{18}{3,3,3,3,3,3} = \frac{18!}{6^6}$

(c) (3 marks) if the cats are indistinguishable and each orphanage must receive at least one cat?

Solution: The answer is the same as the number of ways to distribute 12 indistinguishable cats among 6 orphanages without any further restriction, which is $\binom{12+6-1}{12}$ using the wall trick.

7. (3 marks) Verify and explain the identity $\binom{100}{30} \cdot \binom{70}{25} \cdot \binom{45}{5} = \binom{100}{30,25,5,40}$.

Solution: Verification: $\frac{100!}{30!70!} \frac{70!}{25!45!} \frac{45!}{5!40!} = \frac{100!}{30!25!5!40!}$. Intuitive explanation: I first choose 30 students from 100 to receive A's, then 25 from the remaining 70 students to receive B's, then 5 from the remaining 45 to receive C's and the remaining 40 students get a D. The number of ways to do this is $\binom{100}{30} \cdot \binom{70}{25} \cdot \binom{45}{5}$. On the other hand, I can assign 30 A's, 25 B's, 5 C's and 40 D's all at once, and the number of ways to do this is $\binom{100}{30,25,5,40}$.