July 18, 2014

# Math 302 HW assignment 2. Solutions:

1. 40 percent of Canadians wear a ring on their left hand, 30 percent of them wear a ring on their right hand, and 35 percent of them has no ring at all.

   (a) (2 marks) What is the probability that a random Canadian has rings on both hands?
   *Solution:* Let us pick a random Canadian. Let

   $$L = \{ \text{ wears ring on left hand } \} \quad \text{and} \quad R = \{ \text{ wears ring on right hand } \}.$$

   Then $\mathbf{P}(L) = 0.4$, $\mathbf{P}(R) = 0.3$, $\mathbf{P}((L \cup R)^c) = 0.35$.
   Therefore $\mathbf{P}(L \cup R) = 1 - 0.35 = 0.65$. We know that

   $$0.65 = \mathbf{P}(L \cup R) = \mathbf{P}(L) + \mathbf{P}(R) - \mathbf{P}(L \cap R) = 0.4 + 0.3 - \mathbf{P}(L \cap R)$$

   Thus $\mathbf{P}(L \cap R) = 0.05$, this is the probability that a random Canadian has rings on both hands.

   (b) (2 marks) What percentage of them has a ring on exactly one hand?
   *Solution:* Wears ring only on left hand: $L \setminus R$. Wears ring only on right hand: $R \setminus L$. Wears ring only on exactly one hand: $(L \setminus R)\dot{\cup}(R \setminus L)$.

   $$\mathbf{P}(L \setminus R) = \mathbf{P}(L) - \mathbf{P}(L \cap R) = 0.4 - 0.05 = 0.35,$$
   $$\mathbf{P}(R \setminus L) = \mathbf{P}(R) - \mathbf{P}(L \cap R) = 0.3 - 0.05 = 0.25,$$

   thus $\mathbf{P}[(L \setminus R)\dot{\cup}(R \setminus L)] = \mathbf{P}(L \setminus R) + \mathbf{P}(R \setminus L) = 0.35 + 0.25 = 0.6$.

2. In a class of 30 students, 3 foreign languages are thought: Russian, Polish and Spanish. 9 students attend (at least) Spanish class, 11 students attend (at least) Polish and 13 attend (at least) Russian. 4 attend (at least) Spanish and Russian, 3 attend (at least) Spanish and Polish, 5 attend (at least) Polish and Russian. 2 attend all the three languages.

   (a) (2 marks) What is the prob. that a randomly chosen student learns a Slavic language?
   *Solution*: Polish and Russian are Slavic languages, Spanish is not (it is a Romance language).

   $$\mathbf{P}(P \cup R) = \mathbf{P}(P) + \mathbf{P}(R) - \mathbf{P}(P \cap R) = \frac{11}{30} + \frac{13}{30} - \frac{5}{30} = \frac{19}{30}$$

   (b) (3 marks) How many students learn no foreign language at all?
   *Solution*:

   $$\mathbf{P}(P \cup R \cup S) = \mathbf{P}(P) + \mathbf{P}(R) + \mathbf{P}(S) - \mathbf{P}(P \cap R) - \mathbf{P}(P \cap S) - \mathbf{P}(S \cap R) + \mathbf{P}(P \cap R \cap S) = \frac{11 + 13 + 9 - 5 - 3 - 4 + 2}{30} = \frac{23}{30}.$$

   Thus the number of students who learn no foreign language is $30 - 23 = 7$.

3. A die is rolled four times. What is the probability that the highest number is ...

   (a) (2 marks) at least 4?
   *Solution:* Denote by $X_1, X_2, X_3, X_4$ the outcomes of the four dice rolls and denote by by $X = \max\{X_1, X_2, X_3, X_4\}$ the highest number rolled. First note that for any $1 \leq k \leq 6$

   $$\mathbf{P}(X \leq k) = \mathbf{P}(X_1 \leq k,\, X_2 \leq k,\, X_3 \leq k,\, X_4 \leq k) = \mathbf{P}(X_1 \leq k)\mathbf{P}(X_2 \leq k)\mathbf{P}(X_3 \leq k)\mathbf{P}(X_4 \leq k) = \left(\frac{k}{6}\right)^4.$$

Thus the probability that the highest number is at least 4 is

$$\mathbf{P}(X \geq 4) = 1 - \mathbf{P}(X \leq 3) = 1 - \left(\frac{3}{6}\right)^4.$$

(b) (3 marks) exactly $k$ (where $1 \leq k \leq 6$)?
*Solution*:

$$\mathbf{P}(X = k) = \mathbf{P}(\{X \leq k\} \setminus \{X \leq k-1\}) = \mathbf{P}(X \leq k) - \mathbf{P}(X \leq k-1) = \left(\frac{k}{6}\right)^4 - \left(\frac{k-1}{6}\right)^4.$$

4. One percent of the buses in Vancouver carry one passenger, one percent carries 2 passengers, one percent carries 3 passengers, etc., one percent carries 100 passengers.

(a) (1 mark) What is the prob. that a random bus has more than 50 passengers?
*Solution*: $\frac{1}{2}$.

(b) (4 marks) I call a friend on his cell phone and I find that he is travelling on a bus. What is the probability that his bus has more than 50 passengers?
*Solution*: This is an exercise on the phenomenon called *size-biased sampling.*
Assume that there are 1000 buses in Vancouver. For any $1 \leq k \leq 100$ there are 10 buses with $k$ passengers. Thus for any $1 \leq k \leq 100$ there are $k \cdot 10$ passengers on a bus with exactly $k$ passengers on it. The total number of passengers is

$$N = 10 \cdot (1 + 2 + \cdots + 100) = 10 \cdot \frac{101 \cdot 100}{2}.$$

The total number of passengers on a bus with at most 50 people is

$$M = 10 \cdot (1 + 2 + \cdots + 50) = 10 \cdot \frac{51 \cdot 50}{2}.$$

My friend is a randomly chosen passenger, so the probability that he is on a bus with at most 50 people is

$$\frac{M}{N} = \frac{51 \cdot 50}{101 \cdot 100} = \frac{51}{202}.$$

The probability that his bus has more than 50 passengers is $1 - \frac{51}{202} = \frac{151}{202} \approx 0.75$.

5. An urn contains a black and a white marble. I randomly pick a marble, look at it, put it back and also put a new marble in the urn which has the same colour as the one that I just picked. I repeat this procedure. When there are ...

(a) (2 marks) four marbles in the urn, what is the prob. that two of them are black?
*Solution*: $\frac{1}{3}$, see calculation below.

(b) (2 marks) five marbles in the urn, what is the prob. that three of them are white?
*Solution*: $\frac{1}{4}$, see calculation below.

(c) (3 marks) 100 marbles in the urn, what is the prob. that 38 of them are white?
*Solution*: This exercise is about the so-called *Pólya urn model.*
Denote by $A_k^n$ the event that there are $k$ white marbles in the urn when there are altogether $n$ marbles in the urn, where $n \geq 2$ and $1 \leq k \leq n-1$. We'll prove by induction on $n$ the statement

$$\mathbf{P}(A_k^n) = \frac{1}{n-1} \quad \text{for any} \quad n \geq 2 \quad \text{and} \quad 1 \leq k \leq n-1. \tag{1}$$

The statement (1) is true for $n = 2$, since $\mathbf{P}(A_1^2) = 1$. Assuming that (1) holds for some $n \geq 2$, we'll show that (1) holds for $n+1$ and any $1 \leq k \leq n$, i.e., we'll deduce

$$\mathbf{P}(A_k^{n+1}) = \frac{1}{n} \quad \text{for any} \quad 1 \leq k \leq n. \tag{2}$$

First we show (2) for $k=1$. Note that $A_1^{n+1} \subseteq A_1^n$, thus

$$\mathbf{P}(A_1^{n+1}) = \mathbf{P}(A_1^{n+1} \cap A_1^n) = \mathbf{P}(A_1^{n+1} \,|\, A_1^n)\mathbf{P}(A_1^n) = \frac{n-1}{n} \cdot \frac{1}{n-1} = \frac{1}{n}.$$

Now we show (2) for $k=n$. Note that $A_n^{n+1} \subseteq A_{n-1}^n$, thus

$$\mathbf{P}(A_n^{n+1}) = \mathbf{P}(A_n^{n+1} \cap A_{n-1}^n) = \mathbf{P}(A_n^{n+1} \,|\, A_{n-1}^n)\mathbf{P}(A_{n-1}^n) = \frac{n-1}{n} \cdot \frac{1}{n-1} = \frac{1}{n}.$$

Now we show (2) for $1<k<n$. Note that $A_k^{n+1} \subseteq A_k^n \dot\cup A_{k-1}^n$, thus

$$\mathbf{P}(A_k^{n+1}) = \mathbf{P}[A_k^{n+1} \cap (A_k^n \dot\cup A_{k-1}^n)] = \mathbf{P}[(A_k^{n+1} \cap A_k^n) \dot\cup (A_k^{n+1} \cap A_{k-1}^n)] =$$
$$\mathbf{P}(A_k^{n+1} \cap A_k^n) + \mathbf{P}(A_k^{n+1} \cap A_{k-1}^n) = \mathbf{P}(A_k^{n+1} \,|\, A_k^n)\mathbf{P}(A_k^n) + \mathbf{P}(A_k^{n+1} \,|\, A_{k-1}^n)\mathbf{P}(A_{k-1}^n) =$$
$$\frac{n-k}{n}\frac{1}{n-1} + \frac{k-1}{n}\frac{1}{n-1} = \frac{n-1}{n}\frac{1}{n-1} = \frac{1}{n}.$$

We have checked that (2) holds, so by induction (1) holds. Thus, in particular $\mathbf{P}(A_2^4) = \frac{1}{3}$, $\mathbf{P}(A_3^5) = \frac{1}{4}$ and $\mathbf{P}(A_{38}^{100}) = \frac{1}{99}$.

*Alternative Solution:* Let us prove (1) without induction, using combinatorics.

As we pick the balls from the urn, let us write down the sequence of colours that we picked. For example $BWWBW$ means that first we picked black, then white, then white, then black, then white. We can arrive at the event $A_k^n$ using $\binom{n-2}{k-1}$ different colour sequences of length $n-2$ with $k-1$ whites and $n-k-1$ blacks in it. We claim that every such color sequence occurs with probability $\frac{(k-1)!(n-k-1)!}{(n-1)!}$. We first derive (1) from this claim:

$$\mathbf{P}(A_k^n) = \binom{n-2}{k-1}\frac{(k-1)!(n-k-1)!}{(n-1)!} =$$
$$\frac{(n-2)!}{(k-1)!(n-k-1)!}\frac{(k-1)!(n-k-1)!}{(n-1)!} = \frac{(n-2)!}{(n-1)!} = \frac{1}{n-1}.$$

Now we prove the claim. By the tower property of conditional probabilities, we can write the probability of the occurrence of a colour sequence of length $n-2$ with $k-1$ whites and $n-k-1$ blacks in it as the product of $n-2$ fractions, where each term of the product is the probability that we pick the next colour in the sequence, given the present proportion of blacks and whites determined by the previous picks. For example, the product corresponding to $BWWBW$ is $\frac{1}{2}\frac{1}{3}\frac{2}{4}\frac{2}{5}\frac{3}{6}$. In a colour sequence of length $n-2$ with $k-1$ whites and $n-k-1$ blacks in it, the product of the denominators is $(n-1)!$, the product of the "white " numerators is $(k-1)!$ and the product of the "black" numerators is $(n-k-1)!$.

6. (3 marks) Ulysses on one of his quests arrives to a triple junction. He knows that one of the roads leads to Athens, the second to Mycenae and the third to Sparta, but he doesn't know which road leads to which city. He also knows that the people of Athens only tell the truth one for third of the time, people of Mycenae tell the truth half of the time and that the people of Sparta never lie. He threw a die to decide which road to choose, giving equal chance to all three options. When he arrived to a city he asked the first man what the Pythagorean theorem was and the man replied "$a^2+b^2=c^2$". What is the probability that Ulysses ended up in Athens?

   *Solution*: The person from the unknown city told the truth about the Pythagorean theorem. We know: $\mathbf{P}(A) = \mathbf{P}(M) = \mathbf{P}(S) = \frac{1}{3}$, moreover $\mathbf{P}(T \,|\, A) = \frac{1}{3}$, $\mathbf{P}(T \,|\, M) = \frac{1}{2}$ and $\mathbf{P}(T \,|\, S) = 1$.

$$\mathbf{P}(A \,|\, T) = \frac{\mathbf{P}(A \cap T)}{\mathbf{P}(T)} = \frac{\mathbf{P}(T \,|\, A)\mathbf{P}(A)}{\mathbf{P}(T \,|\, A)\mathbf{P}(A) + \mathbf{P}(T \,|\, M)\mathbf{P}(M) + \mathbf{P}(T \,|\, S)\mathbf{P}(S)} =$$
$$\frac{\frac{1}{3}\cdot\frac{1}{3}}{\frac{1}{3}\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{1}{3} + 1\cdot\frac{1}{3}} = \frac{1/9}{1/9+1/6+1/3} = \frac{2}{2+3+6} = \frac{2}{11}$$

7. Every male customer of an automobile insurance company files a claim every year, independently from the other year, with probability $p_m$. Every female customer files a claim every year independently from the other year, with probability $p_f$, where $p_m \neq p_f$. We know that $\alpha \cdot 100$ percent of the customers are male. Let us track a randomly chosen customer for two consecutive years and denote by $A_i$ the event that the customer files a claim in year $i$, where $i = 1, 2$.

   (a) (4 marks) Which probability is bigger, $\mathbf{P}(A_2)$ or $\mathbf{P}(A_2|A_1)$ ? (calculate both and compare)
   *Solution*:

$$\mathbf{P}(A_2) = \mathbf{P}(A_1) = \mathbf{P}(A_1 \,|\, M)\mathbf{P}(M) + \mathbf{P}(A_1 \,|\, F)\mathbf{P}(F) = p_m \cdot \alpha + p_f \cdot (1-\alpha).$$

$$\mathbf{P}(A_2|A_1) = \frac{\mathbf{P}(A_1 \cap A_2)}{\mathbf{P}(A_1)} = \frac{p_m^2 \cdot \alpha + p_f^2 \cdot (1-\alpha)}{p_m \cdot \alpha + p_f \cdot (1-\alpha)}.$$

   We claim that if $p_m \neq p_f$ and $0 < \alpha < 1$ then $\mathbf{P}(A_2) < \mathbf{P}(A_2|A_1)$. It is enough to show

$$\big(p_m \cdot \alpha + p_f \cdot (1-\alpha)\big)^2 < p_m^2 \cdot \alpha + p_f^2 \cdot (1-\alpha).$$

   Let us define a random variable $X$ with probability mass function

$$f(p_m) = \alpha, \qquad f(p_f) = 1-\alpha.$$

   Then

$$0 < \mathrm{Var}(X) = \mathbf{E}(X^2) - \mathbf{E}(X)^2 = p_m^2 \cdot \alpha + p_f^2 \cdot (1-\alpha) - \big(p_m \cdot \alpha + p_f \cdot (1-\alpha)\big)^2.$$

   (b) (3 marks) If $p_m = 0.015$, $p_f = 0.01$ and $\alpha = 0.6$, what is the probability that the customer is male if we know that he/she filed exactly one claim in these two years?
   *Solution*: We apply Bayes' rule:

$$\mathbf{P}[M \,|\, (A_1 \cap A_2^c)\dot{\cup}(A_1^c \cap A_2)] = \frac{\mathbf{P}[M \cap \big((A_1 \cap A_2^c)\dot{\cup}(A_1^c \cap A_2)\big)]}{\mathbf{P}[(A_1 \cap A_2^c)\dot{\cup}(A_1^c \cap A_2)]} =$$

$$\frac{\alpha 2 p_m(1-p_m)}{\alpha 2 p_m(1-p_m) + (1-\alpha)2p_f(1-p_f)} = \frac{6 \cdot 2 \cdot 15 \cdot 85}{6 \cdot 2 \cdot 15 \cdot 85 + 4 \cdot 2 \cdot 10 \cdot 90}$$