July 18, 2014

Math 302 HW assignment 3 solutions

1. I throw three dice and denote the outcomes by X_1, X_2, X_3. Let us define the events

$$A = \{X_1 \neq X_2\}, \quad B = \{X_2 \neq X_3\}, \quad C = \{X_1 \neq X_3\}.$$

   (a) (3 marks) Calculate $\mathbf{P}(A)$, $\mathbf{P}(A \,|\, B)$ and $\mathbf{P}(A \,|\, B \cap C)$.

   Solution: We calculate probabilities by counting the number of preferable outcomes and divide by the total number of possible outcomes.

   $\mathbf{P}(A) = \frac{6\cdot 5}{6\cdot 6}$ because there are 6 ways to choose X_1 and then 5 ways to choose X_2 such that $X_1 \neq X_2$. Thus $\mathbf{P}(A) = \frac{5}{6}$.

   $\mathbf{P}(A \,|\, B) = \frac{\mathbf{P}(A\cap B)}{\mathbf{P}(B)}$ and $\mathbf{P}(A \cap B) = \frac{6\cdot 5\cdot 5}{6\cdot 6\cdot 6} = \frac{25}{36}$, because there are 6 ways to choose X_2 and then 5 ways to choose X_1 so that $X_1 \neq X_2$ and 5 ways to choose X_3 so that $X_2 \neq X_3$. Also note that $\mathbf{P}(B) = \frac{5}{6}$. Thus $\mathbf{P}(A \,|\, B) = \frac{5}{6}$.

   $\mathbf{P}(A \,|\, B \cap C) = \frac{\mathbf{P}(A\cap B\cap C)}{\mathbf{P}(B\cap C)}$ and $\mathbf{P}(A \cap B \cap C) = \frac{6\cdot 5\cdot 4}{6\cdot 6\cdot 6} = \frac{20}{36}$ because there are 6 ways to choose X_1 and then 5 ways to choose X_2 so that $X_1 \neq X_2$ and then 4 ways to choose X_3 so that $X_1 \neq X_3$ and $X_2 \neq X_3$. Also note that $\mathbf{P}(B \cap C) = \frac{25}{36}$, thus $\mathbf{P}(A \,|\, B \cap C) = \frac{4}{5}$.

   (b) (2 marks) Are B and C independent? Are A, B and C completely independent? Why?

   Solution: A and B are independent, because $\mathbf{P}(A \cap B) = \frac{25}{36} = \mathbf{P}(A)\mathbf{P}(B)$. Similarly A and C are independent, moreover B and C are independent.

   Yet A, B and C are not completely independent, because

$$\mathbf{P}(A \cap B \cap C) = \frac{20}{36} \neq \frac{5^3}{6^3} = \mathbf{P}(A)\mathbf{P}(B)\mathbf{P}(C).$$

2. (2 marks) In a class there are four freshman boys, six freshman girls, six sophomore boys and a certain number of sophomore girls. When a student is selected at random, the following two events are independent:

$$A = \{ \text{ the selected student is a boy } \} \quad \text{and} \quad B = \{ \text{ the selected student is a freshman } \}.$$

   What is the number of sophomore girls?

   Solution: Let N_{sg} denote the number of sophomore girls. The total number of students in the class is $N = 4 + 6 + 6 + N_{sg}$. Our probability space Ω is the set of students. Then A is the set of boys and B is the set of freshmen. $\mathbf{P}(E) = \frac{|E|}{N}$ for any $E \subseteq \Omega$. Let

$$p_b := \mathbf{P}(A), \quad p_g := \mathbf{P}(A^c), \quad p_f := \mathbf{P}(B), \quad p_s := \mathbf{P}(B^c).$$

   We know that A and B are independent, thus A and B^c are also independent, A^c and B are also independent and A^c and B^c are also independent, thus we have

$$\mathbf{P}(A \cap B) = p_b \cdot p_f = \frac{4}{N}, \quad \mathbf{P}(A^c \cap B) = p_g \cdot p_f = \frac{6}{N}, \tag{1}$$

$$\mathbf{P}(A \cap B^c) = p_b \cdot p_s = \frac{6}{N}, \quad \mathbf{P}(A^c \cap B^c) = p_g \cdot p_s = \frac{N_{sg}}{N}. \tag{2}$$

   Thus

$$\frac{4}{6} = \frac{4/N}{6/N} = \frac{p_b \cdot p_f}{p_b \cdot p_s} = \frac{p_f}{p_s} = \frac{p_g \cdot p_f}{p_g \cdot p_s} = \frac{6/N}{N_{sg}/N} = \frac{6}{N_{sg}},$$

   therefore $N_{sg} = 9$.

3. (4 marks) On a roulette wheel there are 18 black and 18 red numbers, plus the number 0 which is green. Before I start I have 100 dollars. My strategy is to keep betting 100 dollars on red again and again until I have either 0 or 1000 dollars. Denote by X my net profit (i.e., the difference between my wealth before starting and after finishing the game). What is $\mathbf{E}(X)$?

Solution: This is a variant of the gambler's ruin problem.

Let p_k denote the probability that I end up with 1000 dollars rather than 0 dollars if I start with $k \cdot 100$ dollars. Then

$$p_0 = 0, \quad p_{10} = 1, \quad p_k = p \cdot p_{k+1} + (1-p) \cdot p_{k-1} \text{ if } k = 1, 2, \dots, 9, \quad p = \frac{18}{37}.$$

Thus for $k = 1, 2, \dots, 9$ we have

$$p \cdot p_k + (1-p) \cdot p_k = p \cdot p_{k+1} + (1-p) \cdot p_{k-1},$$

which can be rearranged as

$$(1-p) \cdot (p_k - p_{k-1}) = p \cdot (p_{k+1} - p_k),$$

thus we have

$$p_{k+1} - p_k = \frac{1-p}{p} \cdot (p_k - p_{k-1}).$$

This can be iterated, thus for any $k = 1, 2, \dots, 9$ we have

$$p_{k+1} - p_k = \frac{1-p}{p} \cdot (p_k - p_{k-1}) = \left(\frac{1-p}{p}\right)^2 \cdot (p_{k-1} - p_{k-2}) = \dots = \left(\frac{1-p}{p}\right)^k \cdot (p_1 - p_0).$$

Therefore for any $K = 1, 2, \dots, 10$ we have

$$p_K = p_K - p_0 = \sum_{k=1}^{K} (p_k - p_{k-1}) = (p_1 - p_0) \sum_{k=1}^{K} \left(\frac{1-p}{p}\right)^{k-1} = (p_1 - p_0) \frac{\left(\frac{1-p}{p}\right)^K - 1}{\frac{1-p}{p} - 1}.$$

Thus for any $K = 1, 2, \dots, 10$ we have

$$p_K = C \cdot \left(\left(\frac{1-p}{p}\right)^K - 1 \right),$$

where C is a constant that does not depend on K. Since $K = 10$ gives $p_{10} = 1$, we must have $C = \left(\left(\frac{1-p}{p}\right)^{10} - 1 \right)^{-1}$, therefore for any $k = 0, 1, 2, \dots, 9, 10$ we have

$$p_k = \frac{\left(\frac{1-p}{p}\right)^k - 1}{\left(\frac{1-p}{p}\right)^{10} - 1}.$$

Thus if I start with \$100 and follow my strategy, my expected net profit in dollars will be

$$\mathbf{E}(X) = 900 \cdot p_1 + (-100) \cdot (1 - p_1) = 1000 \cdot p_1 - 100 =$$

$$1000 \cdot \frac{\frac{1-\frac{18}{37}}{\frac{18}{37}} - 1}{\left(\frac{1-\frac{18}{37}}{\frac{18}{37}}\right)^{10} - 1} - 100 = 1000 \cdot \frac{\frac{19}{18} - 1}{\left(\frac{19}{18}\right)^{10} - 1} - 100 = -22.53$$

4. Two equally good ping pong players (Alice and Bob) compete.

   (a) (2 marks) Assuming that the game goes on indefinitely, let us denote by X the number of rounds they that need to play until Bob wins 6 rounds. What is the distribution of X?

   Solution: Independent Bernoulli trials with probability $1/2$, X is the number of trials until the 6th success, thus $X \sim \text{NBin}(6, \frac{1}{2})$.

   (b) (2 marks) What is $\mathbf{E}(X)$? Why?

   Solution: We have learnt in class that $X = X_1 + \cdots + X_6$ where $X_1, \ldots, X_6$ are independent random variables with distribution $\text{Geo}(1/2)$. Thus

$$\text{E}(X) = \text{E}(X_1 + \cdots + X_6) = \text{E}(X_1) + \cdots + \text{E}(X_6) = 6 \cdot \frac{1}{1/2} = 12.$$

   (c) (3 marks) They compete to see who will be the first to win 11 games. The game gets interrupted when when Bob won 5 rounds and Alice won 8 rounds. What is the fair division of the price money (100 dollars) between Alice and Bob given this information?

   Solution: The fair division of the price money: $p \cdot 100$ dollars go to Bob and $(1-p) \cdot 100$ dollars go to Alice where p is the probability that Bob wins 6 more rounds before Alice wins 3 more rounds. X is the number of rounds that they need to play until Bob wins 6 more rounds, so if $X = 6, 7, 8$ then Bob wins and if $X \geq 9$ then Alice wins, since $X - 6$ is the number of rounds Alice wins before Bob wins his sixth round, and this number has to be 0, 1 or 2 if Bob wins, because Alice is only 3 points short of winning the match. Thus

$$p = \sum_{m=6}^{8} \mathbf{P}(X = m) = \sum_{m=6}^{8} \binom{m-1}{5} \cdot \left(\frac{1}{2}\right)^6 \cdot \left(1 - \frac{1}{2}\right)^{m-6} =$$

$$2^{-6} + 6 \cdot 2^{-7} + \frac{7 \cdot 6}{2} 2^{-8} = \frac{37}{256} = 0.144$$

   Thus \$14.4 goes to Bob, the rest of the price money goes to Alice.

5. In Dragonland the frequency of an n-headed dragon is

$$p_n = \binom{6}{n-1} \cdot 0.7^{n-1} \cdot 0.3^{7-n}, \qquad n = 1, 2, \ldots, 7.$$

   Beheading a dragon is a dangerous business: one succeeds in cutting off each head independently from the others with 90% chance, and if he/she fails to cut it off, then the head eats him/her.

   (a) (2 mark) What is the average number of heads of a dragon from Dragonland?

   Solution: Note that if X denotes the number of heads on a random dragon, then $X = Y + 1$ where $Y \sim \text{Bin}(6, 0.7)$. Indeed, for any $m = 0, 1, \ldots, 6$ we have

$$\mathbf{P}(Y = m) = \binom{6}{m} \cdot 0.7^m \cdot 0.3^{6-m},$$

   thus for any $n = 1, 2, \ldots, 7$ we have

$$\mathbf{P}(X = n) = \mathbf{P}(Y = n - 1) = \binom{6}{n-1} \cdot 0.7^{n-1} \cdot 0.3^{6-(n-1)} = p_n.$$

   Thus the average number of heads on a dragon is

$$\mathbf{E}(X) = \mathbf{E}(Y + 1) = \mathbf{E}(Y) + 1 = 6 \cdot (0.7) + 1 = \frac{52}{10} = 5.2$$

(b) (2 marks) I come across a dragon, but I can't see from the fog how many heads it has. What is my chance of surviving this encounter?

Solution: One can calculate this using the total probability rule and the binomial theorem:

$$\mathbf{P}(\text{survive}) = \sum_{n=1}^{7} \mathbf{P}(\text{survive} \mid n\text{-headed}) \cdot p_n = \sum_{n=1}^{7} (0.9)^n \cdot p_n =$$

$$\sum_{n=1}^{7} (0.9)^n \cdot \binom{6}{n-1} \cdot 0.7^{n-1} \cdot 0.3^{7-n} = 0.9 \cdot \sum_{n=1}^{7} \binom{6}{n-1} \cdot 0.63^{n-1} \cdot 0.3^{7-n} =$$

$$0.9 \cdot \sum_{m=0}^{6} \binom{6}{m} \cdot 0.63^m \cdot 0.3^{6-m} = 0.9 \cdot (0.63 + 0.3)^6 = 0.9 \cdot 0.93^6.$$

But one can also use the representation $X = Y + 1$ where $Y \sim \text{Bin}(6, 0.7)$. This means that every dragon has a main head which is always there plus six more necks and from each of these six necks a head grows out with probability 0.7, independently from the other necks. I defeat each of these necks independently from each others with probability $0.3 + 0.7 \cdot 0.9 = 0.93$, because I defeat a neck if there is no head there (this happens with probability 0.3) or if there is a head and I kill it (this happens with probability $0.7 \cdot 0.9$). Thus I defeat the main head and the six necks with probability $0.9 \cdot 0.93^6$.

(c) (2 marks) After the battle I meet a friend who also just killed a dragon. What is the probability that it was 7-headed?

Solution:

$$\mathbf{P}(\text{7-headed} \mid \text{survive}) = \frac{\mathbf{P}(\text{7-headed and survive})}{\mathbf{P}(\text{survive})} = \frac{p_7 \cdot 0.9^7}{0.9 \cdot 0.93^6} = \frac{0.7^6 \cdot 0.9^7}{0.9 \cdot 0.93^6} = \left(\frac{63}{93}\right)^6$$

6. Alice and Bob play target shooting. With each shot, Alice hits the target with 15% chance, Bob hits it with 10% chance. Bob starts and they take turns. Whoever hits the target first wins.

(a) (3 marks) What is the probability that Alice wins?

Solution: Let X denote the number of bullets shot. If $X = 2k + 1$ for some $k \geq 0$ then Bob wins, if $X = 2k + 2$ for some $k \geq 0$ then Alice wins. Let

$$p_A := 0.15, \qquad p_B := 0.1$$

Then

$$\mathbf{P}(X = 2k+1) = (1 - p_B)^k \cdot (1 - p_A)^k \cdot p_B, \qquad k \geq 0,$$
$$\mathbf{P}(X = 2k+2) = (1 - p_B)^k \cdot (1 - p_A)^k \cdot (1 - p_B) \cdot p_A, \qquad k \geq 0.$$

Thus

$$\mathbf{P}(\text{Alice wins}) = \sum_{k=0}^{\infty} \mathbf{P}(X = 2k+2) = (1 - p_B) \cdot p_A \cdot \sum_{k=0}^{\infty} [(1 - p_B)(1 - p_A)]^k =$$

$$\frac{(1 - p_B) \cdot p_A}{1 - (1 - p_B)(1 - p_A)}$$

(b) (3 marks) What is the expectation of the number of bullets shot throughout the game?

Solution: Let us separate the even numbered and odd numbered terms of the sum:

$$\mathbf{E}(X) = \sum_{n=1}^{\infty} n \cdot \mathbf{P}(X = n) =$$

$$\sum_{k=0}^{\infty} (2k+1) \cdot \mathbf{P}(X = 2k+1) + \sum_{k=0}^{\infty} (2k+2) \cdot \mathbf{P}(X = 2k+2) = \mathcal{B} + \mathcal{A}$$

$$\mathcal{A} = 2(1-p_B)\cdot p_A \cdot \sum_{k=0}^{\infty}(k+1)\cdot[(1-p_B)(1-p_A)]^k = $$
$$2(1-p_B)\cdot p_A \cdot \sum_{n=1}^{\infty} n\cdot[(1-p_B)(1-p_A)]^{n-1} = \frac{2(1-p_B)\cdot p_A}{(1-(1-p_B)(1-p_A))^2}$$

$$\mathcal{B} = \sum_{k=0}^{\infty} 2k\cdot \mathbf{P}(X=2k+1) + \sum_{k=0}^{\infty}\mathbf{P}(X=2k+1) = \mathcal{C}+\mathcal{D}$$

$$\mathcal{C} = 2p_B\cdot\sum_{k=0}^{\infty} k\cdot[(1-p_B)(1-p_A)]^k = 2p_B\cdot\sum_{k=1}^{\infty} k\cdot[(1-p_B)(1-p_A)]^k =$$
$$2p_B\cdot(1-p_B)(1-p_A)\cdot\sum_{k=1}^{\infty} k\cdot[(1-p_B)(1-p_A)]^{k-1} = \frac{2p_B\cdot(1-p_B)(1-p_A)}{(1-(1-p_B)(1-p_A))^2}$$

$$\mathcal{D} = p_B\cdot\sum_{k=0}^{\infty}[(1-p_B)(1-p_A)]^k = \frac{p_B}{1-(1-p_B)(1-p_A)}$$

Putting things together we obtain

$$\mathbf{E}(X) = \mathcal{A}+\mathcal{B} = \mathcal{A}+\mathcal{C}+\mathcal{D} =$$
$$\frac{2(1-p_B)\cdot p_A}{(1-(1-p_B)(1-p_A))^2} + \frac{2p_B\cdot(1-p_B)(1-p_A)}{(1-(1-p_B)(1-p_A))^2} + \frac{p_B}{1-(1-p_B)(1-p_A)} = 8.0851$$

Another Solution (proposed by one of you): Let us group together the kth shot of Bob and the kth shot of Alice and call this a "big round". Let us denote by Y the number of big rounds that they begin before finishing the game. Otherwise stated, let Y denote the number of times Bob shot his gun. Then

$$Y \sim \text{Geo}(p), \qquad p := p_A + p_B - p_A\cdot p_B,$$

because p is the probability that either Bob or Alice succeeds in the first big round, and if both of them fail in the first big round, everything starts all over again. The number of failed big rounds is $Y-1$ and the number of bullets shot in failed big rounds is $2(Y-1)$, thus

$$X = 2(Y-1)+Z$$

where Z is the number of bullets shot in the last big round. Now by the memoryless nature of this game the random variable Z has the same distribution as the number of bullets shot in the first big round, conditioned on the event that either Bob or Alice succeeds in the first big round. In particular, $\{Z=1\}$ and $\{Z=2\}$ are the only possible outcomes, and

$$\mathbf{P}(Z=1) = \frac{p_B}{p_A+p_B-p_A\cdot p_B}, \qquad \mathbf{P}(Z=2) = \frac{(1-p_B)\cdot p_A}{p_A+p_B-p_A\cdot p_B}$$

Note that $\mathbf{P}(Z=2)$ is the probability that Alice wins, as calculated in part (a).

$$\mathbf{E}(X) = \mathbf{E}(2(Y-1)+Z) = 2(\mathbf{E}(Y)-1)+\mathbf{E}(Z) =$$
$$2\left(\frac{1}{p_A+p_B-p_A\cdot p_B}-1\right) + 1\cdot\frac{p_B}{p_A+p_B-p_A\cdot p_B} + 2\cdot\frac{(1-p_B)\cdot p_A}{p_A+p_B-p_A\cdot p_B} = 8.0851$$