July 25, 2014

Math 302 HW assignment 4. Due Friday, August 1 at start of class

Note: Write your student number on each page that you submit. Show all your work! Separate the solutions of different exercises with a line. Draw a frame around your final answer (if it is a number).

1. (2 marks) The probability mass function of the POI(λ) distribution is
$$f(k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$
Calculate the expected value and variance.

2. On a Pacific island beach there is at least one shark attack in a year with probability 1/3.
   (a) (2 marks) What is the probability that there will be at least four shark attacks next year?
   (b) (3 marks) Shark attacks are lethal 75% of the time. What is the probability that there will be no lethal shark attack next year?

3. Find the value of c such that the following functions become probability density functions. Calculate the corresponding cumulative distribution function. Sketch a picture of the graph of the p.d.f. as well as the c.d.f.
   (a) (2 marks) $f(x) = c \cdot (2 - |x|)$ if $-2 \le x \le 2$ and $f(x) = 0$ otherwise.
   (b) (3 marks) $f(x) = c \cdot x^2 e^x$ if $x \le 0$ and $f(x) = 0$ if $x > 0$.
   (c) (2 marks) $f(x) = c \cdot \frac{1}{1+x^2}$ for any x.

4. There are two coffee shops on a street 300 meters from each other. You stroll around the street, and on a uniformly distributed point between the two coffee shops you decide to drink a coffee. You like one coffee shop twice as much as the other one, i.e., you are willing to walk twice as much (but not more) to get your coffee there. You decide between the two coffee shops and walk to one of them to get your coffee. You burn 30 calories per kilometre.
   (a) (2 marks) Calculate the expectation and standard deviation of the calories burnt using the formula $\mathbf{E}(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)\,\mathrm{d}x$ rather than calculating the c.d.f. and the the p.d.f.!
   (b) (2 marks) Calculate the c.d.f. and the p.d.f. of the calories burnt.
   (c) (2 marks) Your coffee has 4.5 calories in it. What is the probability of the event that you burn less calories than what you consume, conditioned on the event that you walk more than 50 meters?

5. How to generate any random variable using a uniform random generator?
   (a) (2 marks) Let $X \sim \text{Unif}[0,1]$ and let $Y = -2\ln(X)$. Calculate the c.d.f. of Y. Now Y is a famous random variable: name it and identify its parameter(s).
   (b) (2 marks) Let $X \sim \text{Unif}[0,1]$ and let $Y = g(X)$. How to choose $g(\cdot)$ if I want $Y \sim \mathcal{N}(0,1)$?

6. (4 marks) Let us define two probability density functions:
$$f_1(x) = \begin{cases} c_1 \cdot x^{-5/2} & \text{if } 1 \le x \le 10^{100} \\ 0 & \text{otherwise} \end{cases} \qquad f_2(x) = \begin{cases} c_2 \cdot x^{-7/2} & \text{if } 1 \le x \le 10^{100} \\ 0 & \text{otherwise} \end{cases}$$
Find c_1 and c_2 so that $f_1(\cdot)$ and $f_2(\cdot)$ are indeed probability density functions (you can omit small errors). Denote by X_1 and X_2 the corresponding random variables. Calculate the corresponding expectations μ_1, μ_2 and variances σ_1, σ_2 numerically (you can omit small errors).
Given this data calculate the lower bound that Chebychev's inequality gives on the probabilities
$$\mathbf{P}(|X_1 - \mu_1| \le 1000) \qquad \text{and} \qquad \mathbf{P}(|X_2 - \mu_2| \le 1000).$$

7. (2 marks) The IQ distribution of students at UBC can be well approximated by a normal distribution with mean 130 and standard deviation 20. In order to design an appropriate final exam, the professor must find the shortest interval $[a, b]$ that contains the IQ of 90% of UBC students. Help the professor with this question.