August 1, 2014

Math 302 HW assignment 5. Due Friday, August 8 at start of class

1. Let us define the joint probability density function $f(x, y)$ of (X, Y) by

$$f(x,y) = \begin{cases} 60x^2y & \text{if } x \geq 0,\ y \geq 0 \text{ and } x + y \leq 1, \\ 0 & \text{otherwise} \end{cases}$$

   (a) (2 marks) Calculate the p.d.f. of X and the p.d.f. Y (i.e., the marginal distributions).

   (b) (2 marks) Are X and Y independent?

   (c) (2 marks) Calculate the covariance of X and Y.

2. Suppose we have a floor made of parallel strips of wood, each with width 3, and we drop a needle of length 1 onto the floor. The aim of this exercise is to find the probability that the needle will lie across a line between two strips.

   (a) (2 marks) Assume that the strips of wood lie in the east-west direction. We call one end of the needle the head and the other end the tip. Let X denote the distance of head from the closest separating line to the south of the head. Let Y denote the angle of the needle measured in radians (let's say $Y = 0$ if the needle is parallel to the lines and the tip points to the east). What is the natural choice of the joint p.d.f. of (X, Y) if we want to model a randomly dropped needle?

   (b) (2 marks) The possible outcomes of (X, Y) form a rectangle. Describe (and draw) the subset A of this rectangle whose points correspond to a needle position that crosses a line!

   (c) (2 marks) Calculate the probability that the needle will lie across a line between two strips.

3. A certain segment of the sky contains Z stars. Each star is either a red giant with probability $1/3$ or a white dwarf with probability $2/3$, independently from the other stars.

   (a) (2 marks) If we condition on the event $\{Z = 4\}$, what is the joint p.m.f. of the red giants X and the white dwarves Y? Fill in a 5×5 table with the $(f(x, y))_{x,y=0}^{4}$ values.

   (b) (3 marks) Show that if $Z \sim \mathrm{POI}(6)$, then the random variables X and Y are independent and have Poisson distribution by calculating the joint p.m.f. of X and Y.

4. The round table of astrologists consists of 144 people. As they examine the laws of the stars, they find that it brings good luck to a Capricorn if a Scorpio sits on his/her right. Calculate the expectation (2 marks) and variance (3 marks) of the number X of lucky Capricorns.

   Hint: Write X as the sum of indicators.

5. Discrete convolution.

   (a) (2 marks) Show that if X and Y are independent integer-valued random variables and $Z = X + Y$ then their p.m.f.'s satisfy $f_Z(z) = \sum_{y=-\infty}^{\infty} f_X(z - y) f_Y(y)$.

   (b) (3 marks) Show that if $X \sim \mathrm{Bin}(n, p)$ and $Y \sim \mathrm{Bin}(m, p)$ then $Z \sim \mathrm{Bin}(m + n, p)$.

6. The aim of this exercise is to show that $-1 \leq \rho(X, Y) \leq 1$ for any pair (X, Y) of random variables, where $\rho(X, Y)$ is the correlation coefficient.

   (a) (1 mark) Express $f(t) := \mathrm{Var}(X + tY) = a \cdot t^2 + b \cdot t + c$ using the bilinearity of covariance.

   (b) (1 mark) Why do we have $b^2 - 4ac \leq 0$? *Hint:* remember the Quadratic Formula.

   (c) (1 mark) Express $\rho(X, Y)$ as a function of a, b, c and show that $-1 \leq \rho(X, Y) \leq 1$.

7. I have an (uncooked) spaghetti of length one, and I break it at a uniform point X. I take the half-spaghetti that is in my left hand, and I break it at a uniform point Y. Find the conditional p.d.f. of Y given X (1 mark), the joint p.d.f. of Y and X (1 mark), the conditional p.d.f. of X given Y (1 mark) and the conditional expectation of X given $Y = 10^{-5}$ (1 mark).