Math 302 HW assignment 5 solutions.

1. Let us define the joint probability density function $f(x,y)$ of (X,Y) by

$$f(x,y) = \begin{cases} 60x^2y & \text{if } x \geq 0,\ y \geq 0 \text{ and } x+y \leq 1, \\ 0 & \text{otherwise} \end{cases}$$

   (a) (2 marks) Calculate the p.d.f. of X and the p.d.f. Y (i.e., the marginal distributions).
   Solution: If $x+y \leq 1$, then $y \leq 1-x$. For any $0 \leq x \leq 1$:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\mathrm{d}y = \int_0^{1-x} 60x^2y\mathrm{d}y = [30x^2y^2]_0^{1-x} = 30x^2(1-x)^2,$$

   thus $f_X(x) = 30x^2(1-x)^2\mathbb{1}[0 \leq x \leq 1]$.
   If $x+y \leq 1$, then $x \leq 1-y$. For any $0 \leq y \leq 1$:

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\mathrm{d}x = \int_0^{1-y} 60x^2y\mathrm{d}x = [20x^3y]_0^{1-y} = 20y(1-y)^3,$$

   thus $f_Y(y) = 20y(1-y)^3\mathbb{1}[0 \leq y \leq 1]$.

   (b) (2 marks) Are X and Y independent?
   Solution: No, because for any $(x,y) \in (0,1)\times(0,1)$ satisfying $x+y > 1$ we have $f_X(x) > 0$, $f_Y(y) > 0$ but $f(x,y) = 0$.

   (c) (2 marks) Calculate the covariance of X and Y.
   Solution:

$$\mathbf{E}(X) = \int_0^1 x \cdot 30x^2(1-x)^2\mathrm{d}x = 1/2$$

$$\mathbf{E}(Y) = \int_0^1 y \cdot 20y(1-y)^3\mathrm{d}x = 1/3$$

$$\mathbf{E}(XY) = \int_0^1 \int_0^{1-y} xy \cdot 60x^2y\,\mathrm{d}x\,\mathrm{d}y = 1/7$$

$$\mathrm{Cov}(X,Y) = \mathbf{E}(XY) - \mathbf{E}(X)\mathbf{E}(Y) = -1/42$$

   Note that it is not surprising that the covariance is negative: the restriction $X+Y \leq 1$ implies that if X is "big", then Y has to be "small", so they have a negative effect on each other.

2. Suppose we have a floor made of parallel strips of wood, each with width 3, and we drop a needle of length 1 onto the floor. The aim of this exercise is to find the probability that the needle will lie across a line between two strips.

   (a) (2 marks) Assume that the strips of wood lie in the east-west direction. We call one end of the needle the head and the other end the tip. Let X denote the distance of head from the closest separating line to the south of the head. Let Y denote the angle of the needle measured in radians (let's say $Y = 0$ if the needle is parallel to the lines and the tip points to the east). What is the natural choice of the joint p.d.f. of (X,Y) if we want to model a randomly dropped needle?
   Solution: This is the famous "Buffon's needle problem" (Google it)
   $0 \leq X \leq 3$, $0 \leq Y \leq 2\pi$, and it is natural to assume that $X \sim \mathrm{Unif}[0,3]$, $Y \sim \mathrm{Unif}[0,2\pi]$ and that (X,Y) are independent. Thus

$$f_X(x) = \frac{1}{3}\mathbb{1}[0 \leq x \leq 3], \quad f_Y(y) = \frac{1}{2\pi}\mathbb{1}[0 \leq y \leq 2\pi],$$

$$f(x,y) = f_X(x)f_Y(y) = \frac{1}{6\pi}\mathbb{1}[0 \leq x \leq 3,\ 0 \leq y \leq 2\pi].$$

(b) (2 marks) The possible outcomes of (X, Y) form a rectangle. Describe (and draw) the subset A of this rectangle whose points correspond to a needle position that crosses a line!

Solution:

If $0 \le X \le 1$ then there is a chance for the needle to intersect the line to the south of the head. The needle crosses that line if $\sin(Y) + X \le 0$ (so Y has to be between π and 2π so that $\sin(Y) < 0$).

If $2 \le X \le 3$ then there is a chance for the needle to intersect the line to the north of the head. The needle crosses that line if $\sin(Y) + X \ge 3$ (so Y has to be between 0 and π so that $\sin(Y) > 0$). Thus, see Figure 1,

$$A = A_1 \cup A_2$$

$$A_1 = \{(x, y) : \pi \le y \le 2\pi,\ 0 \le x \le -\sin(y)\}$$

$$A_2 = \{(x, y) : 0 \le y \le \pi,\ 3 - \sin(y) \le x \le 3\}$$

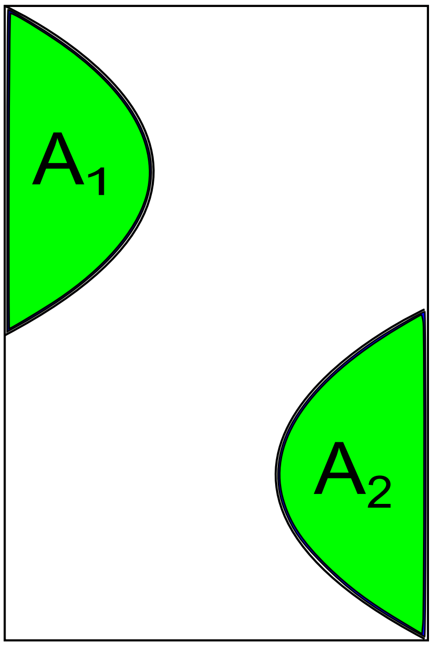

Figure 1: An illustration of $A_1 \cup A_2 = A \subseteq [0, 3] \times [0, 2\pi]$.

(c) (2 marks) Calculate the probability that the needle will lie across a line between two strips.

Solution: Since (X, Y) is uniformly distributed on the rectangle $\Omega = [0, 3] \times [0, 2\pi]$, we have

$$\mathbf{P}[(X, Y) \in A] = \frac{\text{area}(A)}{\text{area}(\Omega)} = \frac{\text{area}(A_1) + \text{area}(A_2)}{6\pi}$$

Now the areas of A_1 and A_2 are both equal to $\int_0^{\pi} \sin(y) \mathrm{d}y = [-\cos(y)]_0^{\pi} = 2$, thus the probability that the needle will lie across a line between two strips is $\frac{2}{3\pi}$.

3. A certain segment of the sky contains Z stars. Each star is either a red giant with probability $1/3$ or a white dwarf with probability $2/3$, independently from the other stars.

(a) (2 marks) If we condition on the event $\{Z = 4\}$, what is the joint p.m.f. of the red giants X and the white dwarves Y? Fill in a 5×5 table with the $(f(x, y))_{x,y=0}^{4}$ values.

Solution:

Conditioned on $\{Z = z\}$, the conditional distribution of red giants is $X \sim \text{Bin}(z, p)$, where $p = \frac{1}{3}$ and the number of white dwarves is just $Y = z - X$. Note that $Y \sim \text{Bin}(z, 1 - p)$. Thus for any $0 \le x \le z$ we have

$$\mathbf{P}(X = x \mid Z = z) = \binom{z}{x} p^x (1 - p)^{z-x} . \tag{1}$$

In particular the entries of the 5×5 table with the $(f(x,y))_{x,y=0}^{4}$ values are:

$$f(x,y) = \begin{cases} \binom{4}{x}\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{4-x} & \text{if } x+y=4, \\ 0 & \text{if } x+y \neq 4. \end{cases}$$

(b) (3 marks) Show that if $Z \sim \mathrm{POI}(6)$, then the random variables X and Y are independent and have Poisson distribution by calculating the joint p.m.f. of X and Y.

Solution: We'll show more generally that if $Z \sim \mathrm{POI}(\lambda)$ and each star is red with probability p or white with probability $1-p$, independently from other stars, then $X \sim \mathrm{POI}(p\lambda)$ and $Y \sim \mathrm{POI}((1-p)\lambda)$, moreover (X,Y) are independent. In order to show all of this it is enough to show that the joint p.m.f. of (X,Y) is

$$f(x,y) = \left(e^{-p\lambda}\frac{(p\lambda)^x}{x!}\right) \cdot \left(e^{-(1-p)\lambda}\frac{((1-p)\lambda)^y}{y!}\right), \qquad x,y = 0,1,2,\dots$$

Let's show this. First note that $Z = X+Y$, so if $X=x$ and $Y=y$, then $Z = x+y$. Thus

$$\begin{aligned} &f(x,y) = \mathbf{P}(X=x,\, Y=y) = \mathbf{P}(X=x,\, Y=y,\, Z=x+y) = \\ &\mathbf{P}(X=x,\, Y=y \,|\, Z=x+y)\mathbf{P}(Z=x+y) = \mathbf{P}(X=x \,|\, Z=x+y)\mathbf{P}(Z=x+y) \overset{(1)}{=} \\ &\binom{x+y}{x}p^x(1-p)^y \cdot e^{-\lambda}\frac{\lambda^{x+y}}{(x+y)!} = \frac{(x+y)!}{x!\cdot y!}p^x(1-p)^y \cdot e^{-p\lambda}e^{-(1-p)\lambda}\frac{\lambda^x\lambda^y}{(x+y)!} = \\ &\frac{1}{x!}\frac{1}{y!}p^x(1-p)^y \cdot e^{-p\lambda}e^{-(1-p)\lambda}\lambda^x\lambda^y = \frac{1}{x!}\frac{1}{y!}(p\lambda)^x((1-p)\lambda)^y \cdot e^{-p\lambda}e^{-(1-p)\lambda} = \\ &\left(e^{-p\lambda}\frac{(p\lambda)^x}{x!}\right) \cdot \left(e^{-(1-p)\lambda}\frac{((1-p)\lambda)^y}{y!}\right) \end{aligned}$$

4. The round table of astrologists consists of 144 people. As they examine the laws of the stars, they find that it brings good luck to a Capricorn if a Scorpio sits on his/her right. Calculate the expectation (2 marks) and variance (3 marks) of the number X of lucky Capricorns.

Solution: We assume that the star signs of the 144 people are completely independent and uniformly distributed over the 12 possible signs.

Let E_i denote the event that the i'th astrologist is a lucky Capricorn, i.e., E_i is the event that the i'th astrologist is a Capricorn and the person sitting on his right is a Scorpio. Let $X_i = \mathbb{1}[E_i]$ be the indicator variable of the event E_i. Then

$$X = \sum_{i=1}^{144} X_i.$$

First note that for any $1 \le i \le 144$ we have

$$\mathbf{P}(E_i) = \frac{1}{12}\cdot\frac{1}{12} = \frac{1}{144}.$$

Thus

$$\mathbf{E}(X) = \mathbf{E}\left(\sum_{i=1}^{144} X_i\right) = \sum_{i=1}^{144}\mathbf{E}(X_i) = \sum_{i=1}^{144}\frac{1}{144} = 1.$$

Nov let's calculate

$$\mathrm{Var}(X) = \mathrm{Cov}(X,X) = \mathrm{Cov}(\sum_{i=1}^{144} X_i, \sum_{j=1}^{144} X_j) = \sum_{i=1}^{144}\sum_{j=1}^{144}\mathrm{Cov}(X_i,X_j).$$

Note that $X_i \cdot X_j = \mathbb{1}[E_i]\mathbb{1}[E_j] = \mathbb{1}[E_i \cap E_j]$ and that the expectation of the indicator of an event is just the probability of that event, thus

$$\mathrm{Cov}(X_i,X_j) = \mathbf{E}(X_i\cdot X_j) - \mathbf{E}(X_i)\cdot\mathbf{E}(X_j) = \mathbf{P}(E_i \cap E_j) - \mathbf{P}(E_i)\mathbf{P}(E_j).$$

If $i=j$ then $\mathbf{P}(E_i \cap E_j) = \mathbf{P}(E_i) = \frac{1}{144}$, thus $\mathrm{Cov}(X_i,X_i) = \frac{1}{144}\left(1 - \frac{1}{144}\right)$.

If i and j are sitting right next to each other then $E_i \cap E_j = \emptyset$, because for both of them to be a lucky Capricorn, both of them would need to be a Capricorn, but the person on the right also has to be a Scorpio, which is impossible. Thus in this case $\mathbf{P}(E_i \cap E_j) = 0$ and $\mathrm{Cov}(X_i, X_j) = -\frac{1}{144^2}$.

If there is at least one person sitting between i and j then E_i and E_j are independent events, because the outcome of the events E_i and E_j depend on the birthdays of two separate pairs of people, whose birthdays are all independent. Therefore $\mathrm{Cov}(X_i, X_j) = 0$.

Now the sum $\sum_{i=1}^{144} \sum_{j=1}^{144} \mathrm{Cov}(X_i, X_j)$ has 144×144 terms. 144 terms are "diagonal", i.e., $i = j$. $2 \cdot 144$ terms correspond to astrologists i and j are sitting right next to each other (the factor 2 is there because j can sit on the left or the right of i). The rest of the terms of the sum are all zeros, thus

$$\mathrm{Var}(X) = 144 \cdot \frac{1}{144}\left(1 - \frac{1}{144}\right) + 2 \cdot 144 \cdot (-\frac{1}{144^2}) = 1 - \frac{3}{144}$$

5. Discrete convolution.

(a) (2 marks) Show that if X and Y are independent integer-valued random variables and $Z = X + Y$ then their p.m.f.'s satisfy $f_Z(z) = \sum_{y=-\infty}^{\infty} f_X(z-y) f_Y(y)$.

Solution:

$$f_Z(z) = \mathbf{P}(Z = z) = \sum_{y=-\infty}^{\infty} \mathbf{P}(Z = z,\, Y = y) = \sum_{y=-\infty}^{\infty} \mathbf{P}(X = z - y,\, Y = y) =$$

$$\sum_{y=-\infty}^{\infty} \mathbf{P}(X = z - y)\mathbf{P}(Y = y) = \sum_{y=-\infty}^{\infty} f_X(z-y) f_Y(y)$$

(b) (3 marks) Show that if $X \sim \mathrm{Bin}(n, p)$ and $Y \sim \mathrm{Bin}(m, p)$ then $Z \sim \mathrm{Bin}(m + n, p)$.

Solution:

$$f_X(x) = \binom{n}{x} p^x (1-p)^{n-x} \mathbb{1}[0 \le x \le n]$$

$$f_Y(y) = \binom{m}{y} p^y (1-p)^{m-y} \mathbb{1}[0 \le y \le m]$$

We want to show that $f_Z(z) = \sum_{y=-\infty}^{\infty} f_X(z-y) f_Y(y)$ satisfies

$$f_Z(z) = \binom{n+m}{z} p^z (1-p)^{n+m-z} \mathbb{1}[0 \le z \le n+m].$$

Note that indeed $\mathbf{P}(0 \le X \le n) = 1$ and $\mathbf{P}(0 \le Y \le m) = 1$ implies $\mathbf{P}(0 \le Z \le n+m) = \mathbf{P}(0 \le X + Y \le n+m) = 1$, thus $f_Z(z) = 0$ if $z < 0$ or $z > n+m$. So we can assume that $0 \le z \le n+m$ and calculate

$$f_Z(z) = \sum_{y=-\infty}^{\infty} f_X(z-y) f_Y(y) =$$

$$\sum_{y=-\infty}^{\infty} \binom{n}{z-y} p^{z-y} (1-p)^{n-z+y} \mathbb{1}[0 \le z-y \le n] \binom{m}{y} p^y (1-p)^{m-y} \mathbb{1}[0 \le y \le m] =$$

$$p^z (1-p)^{n+m-z} \sum_{y=-\infty}^{\infty} \binom{n}{z-y}\binom{m}{y} \mathbb{1}[0 \le z-y \le n,\ 0 \le y \le m] =$$

$$p^z (1-p)^{n+m-z} \sum_{y=z-n}^{m} \binom{n}{z-y}\binom{m}{y} = p^z (1-p)^{n+m-z} \binom{n+m}{z}.$$

The only question that remains is that why did we have $\sum_{y=z-n}^{m} \binom{n}{z-y}\binom{m}{y} = \binom{n+m}{z}$? This is a combinatorial identity: we have $n+m$ numbered (i.e., distinguishable) balls, n is red, m is blue. How many ways are there to pick z balls out of them? The answer is $\binom{n+m}{z}$.

Now let's calculate the number of ways to pick z balls out of them, where the number of blue balls is y. This means we need to choose exactly $z-y$ red balls, thus the number of ways is $\binom{n}{z-y}\binom{m}{y}$ by the multiplication trick. Now y (the number of blue balls picked) can be at most m, and $z-y$ (the number of red balls picked) can be at most n, so $y \leq z-n$. Thus if we sum $\binom{n}{z-y}\binom{m}{y}$ for y values between $z-n$ and m, then we calculated the number of ways to pick z balls out of $n+m$ balls, where the number of blue balls is y, but y can be anything, so we just calculated that the number of ways to pick z balls out of $n+m$ balls is equal to $\sum_{y=z-n}^{m}\binom{n}{z-y}\binom{m}{y}$.

Another Solution: We want to show that $X \sim \mathrm{Bin}(n,p)$ and $Y \sim \mathrm{Bin}(m,p)$ are independent then $X+Y = Z \sim \mathrm{Bin}(m+n,p)$. Now X is the number of successful attempts out of n independent p-trials and Y is the number of successful attempts out of another m independent p-trials, so $Z = X+Y$ together counts the number of successful attempts out of $n+m$ independent p-trials, hence $Z \sim \mathrm{Bin}(m+n,p)$. That's all.

6. The aim of this exercise is to show that $-1 \leq \rho(X,Y) \leq 1$ for any pair (X,Y) of random variables, where $\rho(X,Y)$ is the correlation coefficient.

   (a) (1 mark) Express $f(t) := \mathrm{Var}(X+tY) = a \cdot t^2 + b \cdot t + c$ using the bilinearity of covariance.

   (b) (1 mark) Why do we have $b^2 - 4ac \leq 0$? *Hint:* remember the Quadratic Formula.

   (c) (1 mark) Express $\rho(X,Y)$ as a function of a,b,c and show that $-1 \leq \rho(X,Y) \leq 1$.

   Solution: The solution is explained in Tim Hulshof's typed MATH302 lecture notes (see Principle 31 on page 85), the link to the file is available on my MATH302 lecture notes webpage.

7. I have an (uncooked) spaghetti of length one, and I break it at a uniform point X. I take the half-spaghetti that is in my left hand, and I break it at a uniform point Y. Find the conditional p.d.f. of Y given X (1 mark), the joint p.d.f. of Y and X (1 mark), the conditional p.d.f. of X given Y (1 mark) and the conditional expectation of X given $Y = 10^{-5}$ (1 mark).

   Solution:

   $X \sim \mathrm{Unif}[0,1]$, thus $f_X(x) = \mathbb{1}[0 \leq x \leq 1]$ and conditioned on $\{X = x\}$, $Y \sim \mathrm{Unif}[0,x]$, thus $f_{Y|X}(y|x) = \frac{1}{x}\mathbb{1}[0 \leq y \leq x]$. Now $f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$, which can be rearranged to obtain

$$f(x,y) = f_X(x) f_{Y|X}(y|x) = \frac{1}{x}\mathbb{1}[0 \leq y \leq x \leq 1].$$

   For any $0 \leq y \leq 1$ we have

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\mathrm{d}x = \int_y^1 \frac{1}{x}\mathrm{d}x = -\ln(y).$$

   Thus

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = -\frac{1}{x\ln(y)}\mathbb{1}[y \leq x \leq 1].$$

$$\mathbf{E}(X \mid Y = y) = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)\mathrm{d}x = \int_y^1 -\frac{1}{\ln(y)}\mathrm{d}x = \frac{y-1}{\ln(y)}$$

$$\mathbf{E}(X \mid Y = 10^{-5}) = \frac{10^{-5}-1}{\ln(10^{-5})} \approx \frac{1}{\ln(10^5)} \approx 0.0868$$