THM: LET $X_1, X_2, \ldots$ I.I.D.

ASSUME $-X_r \sim X_r$ (SYMMETRY)

$$F(x) \underset{A}{:=} \mathbb{P}(X_r \le x)$$

ASSUME: $\boxed{\lim_{x\to\infty} x^{\alpha} \cdot (1 - F(x)) \underset{B}{=} b}$ $\quad \alpha \in (0,2)$, $b \in (0,+\infty)$

$$S_n \underset{C}{:=} X_1 + \ldots + X_n$$

$$\boxed{Z_n \underset{D}{:=} S_n / n^{1/\alpha}}$$

THEN $\boxed{\lim_{n\to\infty} \mathbb{E}\left(e^{itZ_n}\right) \underset{E}{=} e^{-c\cdot|t|^{\alpha}}}$

WITH $\boxed{c \underset{F}{:=} 2 b \alpha \cdot \int_0^{\infty} \frac{1-\cos(y)}{y^{\alpha+1}}\, dy}$

WE WILL NOT PROVE THIS.

NOTE THAT A SPECIAL CASE OF THIS THM WAS STATED ON PAGE 148.

THERE WE HAD $\boxed{b \underset{G}{=} \frac{1}{2}}$ (SEE PAGE 149)

REMARK: $M_n \underset{A}{:=} \max\{X_1, \dots, X_n\}$

$$S_n \underset{B}{:=} X_1 + \dots + X_n$$

IF $\lim_{x \to \infty} x^{\alpha} \cdot (1 - F(x)) \underset{C}{=} b \in \mathbb{R}_+$, $\boxed{\alpha \in \mathbb{R}_+}$

THEN $\lim_{n \to \infty} \mathbb{P}\left(\frac{M_n}{n^{1/\alpha}} \le x \right) \underset{D}{=} \lim_{n \to \infty} F(n^{1/\alpha} \cdot x)^n$

$$= \lim_{n \to \infty} \left(1 - \frac{b + \bar{o}(1)}{(n^{1/\alpha} \cdot x)^{\alpha}} \right)^n \underset{E}{=} \begin{cases} e^{-b/x^{\alpha}} & x \ge 0 \\ 0 & x \le 0 \end{cases}$$

(FRÉCHET DISTRIBUTION)

$\boxed{\alpha > 2}$: $Var(X_1) \underset{F}{<} +\infty$, $|S_n| \underset{G}{\asymp} \sqrt{n}$, $M_n \underset{H}{\asymp} n^{1/\alpha}$

THUS $M_n \underset{I}{\ll} |S_n|$, C.L.T. HOLDS

$\boxed{\alpha < 2}$: $M_n \underset{J}{\asymp} n^{1/\alpha} \underset{K}{\gg} \sqrt{n}$, C.L.T. FAILS

$|S_n| \underset{L}{\asymp} n^{1/\alpha} \underset{M}{\gg} \sqrt{n}$ $\quad Var(X_1) \underset{N}{=} +\infty$

$\boxed{\alpha < 1}$: $M_n \underset{O}{\asymp} n^{1/\alpha} \underset{P}{\gg} n$: LAW OF LARGE NUMBERS FAILS

$\mathbb{E}(|X_1|) \underset{Q}{=} +\infty$, $|S_n| \underset{R}{\asymp} n^{1/\alpha} \underset{S}{\gg} n$

DEF: THE HARMONIC MEAN OF $X_1, \ldots, X_n$:

$$\boxed{H_n \underset{A}{:=} \frac{n}{\frac{1}{X_1} + \ldots + \frac{1}{X_n}}}$$

EX: LET $X_1, X_2, \ldots$ I.I.D., P.D.F. $f(x)$

$f(-x) \underset{B}{\equiv} f(x)$, f IS CONTINUOUS, $f(0) > 0$

THEN $\boxed{H_n \underset{C}{\Rightarrow} CAU\left(0, \frac{1}{\pi \cdot f(0)}\right)}$

SOLUTION: LET $\boxed{Y_k \underset{D}{:=} \frac{1}{X_k}}$, $\boxed{-Y_k \underset{E}{\sim} Y_k}$

LET $F(x) \underset{F}{=} \mathbb{P}(Y_k \leq x)$

THEN $\lim\limits_{x \to \infty} x \cdot (1 - F(x)) \underset{G}{=} \lim\limits_{x \to \infty} x \cdot \mathbb{P}(Y_k > x) \underset{H}{=}$

$= \lim\limits_{x \to \infty} x \cdot \mathbb{P}\left(\frac{1}{X_k} > x\right) \underset{I}{=} \lim\limits_{x \to \infty} x \cdot \mathbb{P}\left(0 < X_k < \frac{1}{x}\right)$

$\underset{J}{=} \lim\limits_{\varepsilon \to 0+} \frac{1}{\varepsilon} \cdot \mathbb{P}(0 < X_k < \varepsilon) \underset{K}{=} \lim\limits_{\varepsilon \to 0+} \frac{1}{\varepsilon} \int_0^{\varepsilon} f(y)\,dy \underset{L}{=}$

$= f(0)$

THUS $\boxed{\alpha = 1}$ AND $\boxed{\gamma = f(0)}$ IN THE THM. STATED ON PAGE 164:

IF $S_n \underset{A}{=} Y_1 + \dots + Y_n$, $Z_n \underset{B}{=} S_n / n$

THEN $$\boxed{\lim_{n\to\infty} \mathbb{E}\left(e^{itZ_n}\right) \underset{C}{=} e^{-c\cdot|t|}}$$

WITH $c \underset{D}{=} 2\cdot f(0)\cdot \int_0^\infty \frac{1-\cos(y)}{y^2}\,dy \underset{E}{=}$

SEE PAGE 113

$= f(0)\cdot\int_{-\infty}^{\infty} \frac{1-\cos(y)}{y^2}\,dy \underset{F}{=} f(0)\cdot\pi$

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NOW $\boxed{e^{-f(0)\cdot\pi\cdot|t|} \underset{G}{=} \mathbb{E}\left(e^{itZ_1}\right)}$, $\boxed{Z \underset{H}{\sim} CAU(f(0)\cdot\pi)}$

NOTE: $\boxed{H_n \underset{I}{=} \frac{1}{Z_n}}$ $\boxed{Z_n \underset{J}{\Rightarrow} Z_1}$

NOTE: IF $W \underset{K}{\sim} CAU(1)$ THEN $\frac{1}{W} \underset{L}{\sim} CAU(1)$

PROOF: P.D.F. OF W IS $g_W(x) \underset{M}{=} \frac{1}{\pi}\cdot\frac{1}{1+x^2}$

P.D.F. OF $\frac{1}{W}$ IS $\frac{1}{x^2}\cdot g_W\left(\frac{1}{x}\right) \underset{N}{=}$

$$\frac{1}{x^2}\frac{1}{\pi}\cdot\frac{1}{1+\frac{1}{x^2}} \underset{O}{=} \frac{1}{\pi}\frac{1}{1+x^2} \underset{P}{=} g_W(x)$$ ✓

THUS $\mathbb{P}\left(\frac{1}{W} \geq x\right) \underset{A}{=} \mathbb{P}(W \geq x) \quad \forall x \in \mathbb{R}$

<u>NOTE</u>: $\boxed{-H_n \underset{B}{\sim} H_n}$ AND IF $\boxed{x \geq 0}$ THEN

$$\lim_{n\to\infty} \mathbb{P}(H_n \leq x) \underset{C}{=} \lim_{n\to\infty} \mathbb{P}\left(\frac{1}{Z_n} \leq x\right) \underset{D}{=}$$

$$\lim_{n\to\infty} \left(\mathbb{P}\left(\frac{1}{Z_n} < 0\right) + \mathbb{P}\left(0 < \frac{1}{Z_n} \leq x\right)\right) \underset{E}{=}$$

$$\frac{1}{2} + \lim_{n\to\infty} \mathbb{P}\left(\frac{1}{x} \leq Z_n\right) \underset{F}{=} \frac{1}{2} + \mathbb{P}\left(\frac{1}{x} \leq Z\right) \underset{G}{=}$$

$$= \frac{1}{2} + \mathbb{P}\left(\frac{1}{x} \leq f(0) \cdot \pi \cdot W\right) \underset{H}{=}$$

$$= \frac{1}{2} + \mathbb{P}\left(\frac{1}{x \cdot f(0) \cdot \pi} \leq W\right) \underset{I}{=}$$

$$= \frac{1}{2} + \mathbb{P}\left(\frac{1}{x \cdot f(0) \cdot \pi} \leq \frac{1}{W}\right) \underset{J}{=}$$

$$= \frac{1}{2} + \mathbb{P}\left(0 < W \leq x \cdot f(0) \cdot \pi\right) \underset{K}{=}$$

$$= \mathbb{P}\left(W \leq x \cdot f(0) \cdot \pi\right) \underset{L}{=} \mathbb{P}\left(\frac{W}{\pi \cdot f(0)} \leq x\right)$$

THUS $\boxed{H_n \underset{M}{\Longrightarrow} \frac{W}{\pi \cdot f(0)} \underset{N}{\sim} CAU\left(\frac{1}{\pi \cdot f(0)}\right)}$

✓✓✓

NON-SYMMETRIC STABLE LAWS:

THM: IF φ IS THE CHAR. FUNCTION OF A STABLE DISTRIBUTION, THEN

$\boxed{\alpha \neq 1}$:

$$\varphi(t) = \exp\left(i b t - c \cdot |t|^{\alpha} \cdot \left(1 - i \cdot \operatorname{sgn}(t) \cdot \kappa \cdot \tan\left(\frac{\alpha \pi}{2}\right)\right)\right)$$

A

$\boxed{\alpha = 1}$:

$$\varphi(t) = \exp\left(i b t - c \cdot |t| \cdot \left(1 + i \cdot \operatorname{sgn}(t) \cdot \kappa \cdot \frac{2 \ln(|t|)}{\pi}\right)\right)$$

B

$\boxed{\alpha \in (0,2]}$: INDEX

$\boxed{\kappa \in [-1,1]}$: SKEWNESS

IMPORTANT

$\boxed{c \in (0,+\infty)}$: SCALE

$\boxed{b \in \mathbb{R}}$: SHIFT

NOT IMPORTANT (CAN BE CHANGED BY SHIFTING & SCALING)

W.L.O.G.: $\boxed{c=1}$ $\boxed{b=0}$

NOTATION: $\operatorname{STAB}(\alpha, \kappa, c, b)$

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REMARKS:

1. SYMMETRIC STABLE LAWS: $\boxed{\mathcal{K} = b = 0}$

2. $\boxed{\alpha = 2}$: $\tan\left(\frac{\alpha \cdot \pi}{2}\right) = \tan(\pi) = 0$, THUS

NO SKEWNESS FOR $\alpha = 2$ (NORMAL)

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3. STAB $\left(\alpha = \frac{1}{2}, \mathcal{K} = 1, C = 1, b = 0\right)$ IS LÉVY DISTRIBUTION! INDEED:

$$\varphi(t) \overset{A}{=} \exp\left(-|t|^{1/2} \cdot \left(1 - i \cdot \operatorname{sgn}(t) \cdot \tan\left(\frac{\pi}{4}\right)\right)\right)$$

$$\overset{B}{=} \exp\left(-|t|^{1/2} \cdot \left(1 - i \cdot \operatorname{sgn}(t)\right)\right) \overset{C}{=}$$

$\boxed{(1-i)^2 = -2i}$

$$= \exp\left(-\sqrt{-2it}\right)$$

CHAR. FUNCTION OF LÉVY BY HW 7.2

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4. STAB $\left(\alpha = \frac{1}{2}, \mathcal{K}, C = 1, b = 0\right)$, $\mathcal{K} \in [-1, 1]$

CAN BE CONSTRUCTED AS

$p \cdot X - q \cdot Y$ FOR SOME $p, q \in [0, +\infty)$

WHERE X, Y I.I.D. LÉVY

PROOF: NEXT PAGE

PROOF:

$$\mathbb{E}\left(e^{it\cdot(p\cdot X-q\cdot Y)}\right) \underset{A}{=} \varphi(t\cdot p)\cdot\varphi(-t\cdot q) \underset{B}{=}$$

$$\exp\left(-(|t|\cdot p)^{1/2}\cdot\left(1-i\cdot\operatorname{sgn}(tp)\right)-(|-t|\cdot q)^{1/2}\cdot\left(1-i\cdot\operatorname{sgn}(-tq)\right)\right)$$

$$\underset{C}{=} \exp\left(-|t|^{1/2}\cdot\left(\underbrace{\left(\sqrt{p}+\sqrt{q}\right)}_{\underset{D}{=}\,1}-i\cdot\operatorname{sgn}(t)\cdot\underbrace{\left(\sqrt{p}-\sqrt{q}\right)}_{\underset{E}{=}\,\varkappa}\right)\right)$$

$\leftarrow$ WANT $\rightarrow$

THUS $\boxed{p \underset{F}{:=} \left(\frac{1+\varkappa}{2}\right)^2}$ $\boxed{q \underset{G}{:=} \left(\frac{1-\varkappa}{2}\right)^2}$

DEF: THE FUNCTION $L:(0,+\infty)\to(0,+\infty)$ IS SLOWLY VARYING IF

$$\forall a>0: \quad \boxed{\lim_{x\to\infty}\frac{L(a\cdot x)}{L(x)} \underset{H}{=} 1}$$

THM (THE ULTIMATE STABLE LIMIT THM)

LET $X_1, X_2, \dots$ I.I.D. ASSUME:

(i) $\mathbb{P}(|X_k| > x) \underset{A}{=} x^{-\alpha} \cdot L(x), \quad \alpha \in (0,2)$

L IS SLOWLY VARYING

(ii) $$\lim_{x \to \infty} \frac{\mathbb{P}(X_j > x)}{\mathbb{P}(X_j < -x)} \underset{B}{=} \frac{1+\kappa}{1-\kappa}, \quad \kappa \in [-1,1]$$

LET $a_n \underset{C}{:=} \inf \left\{ x : \mathbb{P}(|X_k| > x) < \frac{1}{n} \right\}$

$b_n \underset{D}{:=} n \cdot \mathbb{E}\left(X_k \cdot \mathbb{1}[|X_k| \le a_n] \right)$

THEN $$\boxed{\frac{S_n - b_n}{a_n} \underset{E}{\Longrightarrow} \mathrm{STAB}(\alpha, \kappa, c, b)}$$

WITH SOME $c \in (0, +\infty)$, $b \in \mathbb{R}$

EX: P.D.F. OF X_k: $$f(x) \underset{F}{:=} \underbrace{\frac{1 + \mathrm{sgn}(x) \cdot \kappa}{2} \cdot \frac{\alpha}{|x|^{\alpha+1}}}_{\text{IF } |x| \ge 1}$$

$$\left.\begin{aligned} \mathbb{P}(X_k > x) &\underset{G}{=} \frac{1+\kappa}{2} \cdot |x|^{-\alpha} \\ \mathbb{P}(X_k < -x) &\underset{H}{=} \frac{1-\kappa}{2} \cdot |x|^{-\alpha} \end{aligned}\right\} \text{IF } |x| \ge 1$$

$$\mathbb{P}(|X_r| > x) \underset{A}{=} |x|^{-\alpha}$$

$$a_n \underset{B}{=} \inf\left\{ x : \mathbb{P}(|X_r| > x) < \frac{1}{n} \right\} = n^{1/\alpha}$$

$$b_n \underset{C}{=} n \cdot \mathbb{E}\left(X_r \cdot \mathbb{1}\left[|X_r| \leq a_n \right] \right) \underset{D}{=}$$

$$n \cdot \int_{-n^{1/\alpha}}^{n^{1/\alpha}} x \cdot f(x)\, dx \underset{E}{=} n \cdot \int_{1}^{n^{1/\alpha}} x \cdot \left(f(x) - f(-x) \right) dx \underset{F}{=}$$

$$= n \cdot \int_{1}^{n^{1/\alpha}} x \cdot \frac{\alpha}{x^{\alpha}}\, dx \underset{G}{\asymp} \begin{cases} n & \text{IF } \alpha > 1 \\ n \cdot \log(n) & \text{IF } \alpha = 1 \\ n^{1/\alpha} & \text{IF } \alpha < 1 \end{cases}$$