Stoch. Proc. HW assignment 10. Due Friday, November 17 at start of class

Note: Each of the 3 questions is worth 10 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises with a horizontal line. Highlight the final answer. Paper format is preferred, but if you submit your homework electronically, pdf format is preferred.

1. There are six machines and six repairmen in a factory. Each machine works for an EXP(2) time interval and then it breaks down. Machines break down independently from one another. If a machine brakes down, a repairmen immediately starts repairing it. The time it takes to repair a machine is EXP(3) distributed. The working times and the repair times are independent. If a machine is fixed, it immediately starts working again, and it works until it breaks down again, etc. Initially all machines work. Denote by X_t the number of working machines at time t.

   (a) Write down the infinitesimal generator of the Markov chain (X_t).

   (b) Find the distribution of X_t. In other words, find $p_t(6, x)$ for all $x = 0, 1, \dots, 6$.
   Hint: First try to solve the analogous question for one single machine and then use a trick that you have already seen in class.

2. *Forest fire model.* We consider the time evolution of a random graph on three vertices. The edge set changes as time evolves. The dynamics is driven by six independent Poisson point processes:

   - for each unordered pair of vertices there is an „edge" PPP with rate 1/3 and upon each arrival of an edge process, we draw and edge between that pair of vertices (unless there is already an edge between them, in which case we don't do anything).
   - for each vertex there is a „lightning" PPP with rate 1 and upon each arrival of a lightning process, a lightning strikes that vertex and immediately burns all of the edges of the connected component of that vertex.

   Denote by G_t the edge set of the graph at time t and let X_t be number of connected components in G_t.

   (a) Argue that (X_t) is a Markov process and find its infinitesimal generator matrix.

   (b) Find the stationary distribution of (X_t).

3. There is an amoeba at time 0 in a Petri dish. An amoeba splits after an EXP(1) distributed waiting time into two amoebae which are identical to the original one. The goal of this exercise is to show that at time t the number X_t of amoebae in the Petri dish has optimistic GEO(e^{-t}) distribution.

   (a) Write down the infinitesimal generator matrix of the continuous-time Markov chain (X_t).

   (b) Write down the *Kolmogorov forward equations* for the family of functions $t \mapsto p_t(1, x), x \in \mathbb{N}$.

   (c) Write down the conjectured value of $p_t(1, x)$ for any $t \in \mathbb{R}_+$ and $x \in \mathbb{N}$.

   (d) Verify that the conjectured functions $t \mapsto p_t(1, x), x \in \mathbb{N}$ indeed satisfy the Kolmogorov forward differential equations and the initial condition $p_0(1, x) = \mathbb{1}[x = 1]$.