Stoch. Proc. HW assignment 11. Due Friday, November 27 at start of class

Note: Each of the 3 questions is worth 10 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises with a horizontal line. Highlight the final answer. Paper format is preferred, but if you submit your homework electronically, pdf format is preferred.

1. Let us consider the following model for the spreading of a computer virus. At all times, each infected computer manages to infect a new computer after an $\mathrm{EXP}(a)$ distributed time. For each infected computer, the virus is removed from that computer after an $\mathrm{EXP}(b)$ distributed time. On the top of this, the hacker who created the virus infects new computers with the virus according to the arrival times of a Poisson point process with rate c. Denote by X_t the number of infected computers at time t. The goal of this exercise is to decide (given the positive real parameters a, b and c) whether this Markov chain is positive recurrent, null recurrent or transient. Loosely speaking, transience means a worldwide epidemic, while recurrence means that the virus can be safely controlled.

   (a) Write down the infinitesimal generator of (X_t).

   (b) Explain why it is OK to assume $a = 1$ without loss of generality if we want to decide about whether this Markov chain is positive recurrent, null recurrent or transient.

   (c) Assuming $a = 1$, find the values of (b, c) for which (X_t) is positive recurrent/null recurrent/transient.
   Hint: The $b = 1$ case is the hardest. When $b = 1$, you will find thus formula useful:

$$1 + a_n = \exp\left(a_n + \mathcal{O}(a_n^2)\right),\ a_n \ll 1.$$

2. Consider the continuous-time Markov chain with state space $S = \{1, 2, 3, 4\}$ and inf. gen. matrix

$$\underline{\underline{G}} = \begin{pmatrix} -4 & 1 & 1 & 2 \\ 2 & -5 & 3 & 0 \\ 0 & 2 & -3 & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix}$$

   (a) Assuming we start from state 1, what is the probability that we reach state 3 before state 4?

   (b) Assuming we start from state 1, what is the expected hitting time of the set $\{3, 4\}$?
   Hint: For both (a) and (b), use the method of *embedded discrete-time Markov chain*, c.f. the Corollary on page 148 of the scanned lecture notes and also page 161.

3. Consider a barbershop with one barber and three chairs: one chair for cutting hair and two other chairs for the customers waiting in line. Time is measured in hours. The distribution of the time it takes to cut the hair of one customer is $\mathrm{EXP}(3)$. Customers arrive according to a Poisson point process with rate 1. If all three chairs are occupied and a new costumer arrives, she leaves immediately without sitting down.

   (a) Find the fraction of potential customers that get their hair cut in this barbershop.

   (b) If currently there are three customers in the barbershop, what is the expected time until the barber can take a break?

   (c) The barbershop has already been operating for a long time. I enter the barbershop: what is the expected time that I spend with waiting for my hair to be cut?