Stoch. Proc. HW assignment 11. Due Friday, November 27 at start of class

1. Let us consider the following model for the spreading of a computer virus. At all times, each infected computer manages to infect a new computer after an EXP(a) distributed time. For each infected computer, the virus is removed from that computer after an EXP(b) distributed time. On the top of this, the hacker who created the virus infects new computers with the virus according to the arrival times of a Poisson point process with rate c. Denote by X_t the number of infected computers at time t. The goal of this exercise is to decide (given the positive real parameters a, b and c) whether this Markov chain is positive recurrent, null recurrent or transient. Loosely speaking, transience means a worldwide epidemic, while recurrence means that the virus can be safely controlled.

   (a) Write down the infinitesimal generator of (X_t).

   (b) Explain why it is OK to assume $a = 1$ without loss of generality if we want to decide about whether this Markov chain is positive recurrent, null recurrent or transient.

   (c) Assuming $a = 1$, find the values of (b, c) for which (X_t) is positive recurrent/null recurrent/transient.

   Solution:

   (a) (X_t) is a birth and death process with birth rates $\lambda_x = c + ax$ and death rates $\mu_x = bx$ for any $x = 0, 1, 2, \ldots$ For the infinitesimal generator, see page 157 of scanned lecture notes.

   (b) If we change the unit of measurement of time and declare the new unit of time length to be $1/a$ then birth rates will be $\widetilde{\lambda}_x = \widetilde{c} + \widetilde{a}x$ and the death rates will be $\widetilde{\mu}_x = \widetilde{b}x$ for any $x = 0, 1, 2, \ldots,$ where $\widetilde{a} = 1$, $\widetilde{b} = b/a$, and $\widetilde{c} = c/a$. The time-changed process is recurrent if and only if the original process is recurrent, since the embedded discrete time Markov chain remains the same. The time-changed process is positive recurrent if and only if the original process is positive recurrent, since the stationary distribution (if it exists) remains the same. So it is OK to assume that $a = 1$ without loss of generality.

   (c) First we show that if $b > 1$ then the birth and death chain is positive recurrent. It is enough to show that $\sum_{x=0}^{\infty} \frac{\lambda_0\lambda_1\ldots\lambda_{x-1}}{\mu_1\mu_2\ldots\mu_x} < +\infty$ (see page 158 of the scanned lecture notes). We will do this using the *ratio test*. If we let $a_x = \frac{\lambda_0\lambda_1\ldots\lambda_{x-1}}{\mu_1\mu_2\ldots\mu_x}$, then $a_{x+1}/a_x = \lambda_x/\mu_{x+1} = \frac{x+c}{b(x+1)}$, and $\lim_{x\to\infty} a_{x+1}/a_x = 1/b < 1$, thus by the ratio test we have $\sum_{x=0}^{\infty} a_x < \infty$.

   Next we show that if $b < 1$ then the birth and death chain is transient. It is enough to show that $\sum_{x=0}^{\infty} \frac{\mu_1\mu_2\ldots\mu_x}{\lambda_1\lambda_1\ldots\lambda_x} < +\infty$ (see page 162 of the scanned lecture notes). This is easy using the ratio test, so we omit the details.

   When $b = 1$ then the ratio test is inconclusive, so we need more sophisticated methods to decide about the convergence/divergence of infinite sums. We have $\frac{\lambda_{y-1}}{\mu_y} = \frac{y-1+c}{y} = 1 + \frac{c-1}{y} = \exp\left(\frac{c-1}{y} + \mathcal{O}(y^{-2})\right)$ for any $y \geq 1$. We will use this to estimate the sum deciding about positive/null recurrence of the birth/death process (we omit the term of the sum corresponding to $x = 0$ as it does not change the convergence/divergence of the sum):

$$\sum_{x=1}^{\infty} \frac{\lambda_0\lambda_1\ldots\lambda_{x-1}}{\mu_1\mu_2\ldots\mu_x} = \sum_{x=1}^{\infty} \exp\left(\sum_{y=1}^{x}\left(\frac{c-1}{y} + \mathcal{O}(y^{-2})\right)\right) = \sum_{x=1}^{\infty} e^{(c-1)\ln(x)+\mathcal{O}(1)} \asymp \sum_{x=1}^{\infty} x^{c-1},$$

   therefore the sum is divergent for any $c > 0$. We can conclude that (X_t) is not positive recurrent if $b = 1$. Now let us estimate the sum deciding about recurrence/transience of the birth/death process when $b = 1$. Note that $\frac{\mu_y}{\lambda_y} = \frac{y}{c+y} = \exp\left(-\frac{c}{y} + \mathcal{O}(y^{-2})\right)$ if $y \geq 1$, therefore (again omitting the term corresponding to $x = 0$) we have

$$\sum_{x=1}^{\infty} \frac{\mu_1\mu_2\ldots\mu_x}{\lambda_1\lambda_1\ldots\lambda_x} = \sum_{x=1}^{\infty} \exp\left(\sum_{y=1}^{x}\left(-\frac{c}{y} + \mathcal{O}(y^{-2})\right)\right) = \sum_{x=1}^{\infty} e^{-c\ln(x)+\mathcal{O}(1)} \asymp \sum_{x=1}^{\infty} x^{-c},$$

   therefore the sum is divergent if $0 < c \leq 1$, but convergent if $c > 1$. We can conclude that (X_t) is null recurrent if $a = b = 1$ and $0 < c \leq 1$, but (X_t) is transient if $a = b = 1$ and $c > 1$.

Remark: If (X_t) is a birth-death chain, then recurrence is equivalent to $\liminf_{t\to\infty} X_t = 0$ (since the process returns to state 0 infinitely often) while transience is equivalent to $\liminf_{t\to\infty} X_t = +\infty$ (since the total time that the process spends in each state is finite).

2. Consider the continuous-time Markov chain with state space $S = \{1, 2, 3, 4\}$ and inf. gen. matrix

$$\underline{\underline{G}} = \begin{pmatrix} -4 & 1 & 1 & 2 \\ 2 & -5 & 3 & 0 \\ 0 & 2 & -3 & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix}$$

(a) Assuming we start from state 1, what is the probability that we reach state 3 before state 4?

(b) Assuming we start from state 1, what is the expected hitting time of the set $\{3, 4\}$?

Solution:

(a) Let $h(x) = P_x(T_3 < T_4)$ for $x = 1, 2$. In words: $h(x)$ is the probability of reaching state 3 before reaching state 4 if we start the process from state x. We want to find $h(1)$. We have

$$h(1) = \frac{1}{4}h(2) + \frac{1}{4} \cdot 1 + \frac{2}{4} \cdot 0, \qquad h(2) = \frac{2}{5}h(1) + \frac{3}{5} \cdot 1 + 0 \cdot 0,$$

because (by page 148 of the scanned lecture notes) if we are in state 1, then we jump to state 2 with probability 1/4, we jump to state 3 with probability 1/4 and we jump to state 4 with probability 2/4, while if we start from state 2, then we jump to state 1 with probability 2/5 and we jump to state 3 with probability 3/5.

Solving this system of linear equations, we obtain $h(1) = \frac{4}{9}$ and $h(2) = \frac{7}{9}$.

(b) Let $f(x) = E_x(T_{\{3,4\}})$ for $x = 1, 2$. In words: $f(x)$ is the expected time until hitting the set $\{3, 4\}$ if we start from state x. We want to find $f(1)$. We have

$$f(1) = \frac{1}{4} + \frac{1}{4}f(2) + \left(\frac{1}{4} + \frac{2}{4}\right) \cdot 0, \qquad f(2) = \frac{1}{5} + \frac{2}{5}f(1) + \left(\frac{3}{5} + 0\right) \cdot 0,$$

because (by page 148 of the scanned lecture notes) if we are in state 1, then the time until the next jump has EXP(4) distribution (hence the expected time until the next jump is 1/4), moreover we jump to state 2 with probability 1/4 and we jump to the set $\{3, 4\}$ with probability $1/4 + 2/4$; and if we are in state 2, then the time until the next jump has EXP(5) distribution (hence the expected time until the next jump is 1/5), moreover we jump to state 1 with probability 2/5 and we jump to the set $\{3, 4\}$ with probability $3/5 + 0$.

Solving this system of linear equations, we obtain $f(1) = 1/3$ and $f(2) = 1/3$.

3. Consider a barbershop with one barber and three chairs: one chair for cutting hair and two other chairs for the customers waiting in line. Time is measured in hours. The distribution of the time it takes to cut the hair of one customer is EXP(3). Customers arrive according to a Poisson point process with rate 1. If all three chairs are occupied and a new costumer arrives, she leaves immediately without sitting down.

   (a) Find the fraction of potential customers that get their hair cut in this barbershop.

   (b) If currently there are three customers in the barbershop, what is the expected time until the barber can take a break?

   (c) The barbershop has already been operating for a long time. I enter the barbershop: what is the expected time that I spend with waiting for my hair to be cut?

Solution:

(a) There are (at least) two different ways of answering this question.
First solution: let us denote by X_t the number of customers in the barbershop at time t. Then the process (X_t) is a continuous time birth and death chain with state space $S = \{0,1,2,3\}$, birth (arrival) rates $\lambda_0 = \lambda_1 = \lambda_2 = 1$ and death (departure) rates $\mu_1 = \mu_2 = \mu_3 = 3$. Let us find the stationary distribution π of (X_t). By the detailed balance condition we have $\pi(x)\lambda_x = \pi(x+1)\mu_{x+1}$ for $x = 0,1,2$. Thus $\pi(0)\cdot 1 = \pi(1)\cdot 3$, that is $\pi(1) = \frac{1}{3}\pi(0)$. Similarly $\pi(2) = \frac{1}{3}\pi(1) = \frac{1}{9}\pi(0)$ and $\pi(3) = \frac{1}{3}\pi(2) = \frac{1}{27}\pi(0)$. Thus $\pi(0) = \left(1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}\right)^{-1} = \frac{27}{27+9+3+1} = \frac{27}{40}$, and $\pi(1) = \frac{9}{40}$, $\pi(2) = \frac{3}{40}$ and $\pi(3) = \frac{1}{40}$. Now if we look at (X_t) for 10^9 hours then the length of time when three chairs are occupied is roughly $\pi(3)\cdot 10^9$. All customers that arrive when three chairs are occupied go away immediately. Customers arrive at rate 1 per hour, so the total number of customers that arrived is roughly 10^9 and the number of customers that go away immediately is roughly equal to the length of time when three chairs are occupied, roughly $\pi(3)\cdot 10^9$. In conclusion, the fraction of potential customers that get their hair cut in this barbershop is $\frac{39}{40}$.
Second solution: Denote by N_t^A the counting process of a PPP with rate 1 and denote by N_t^D the counting process of an independent PPP with rate 3. Denote by $N_t = N_t^A + N_t^D$ the counting process of the merged PPP, which has rate 4. Consider the continuous-time Markov chain (X_t) which jumps to the right if there is arrival in the N_t^A process and jumps to the left if there is arrival in the N_t^D process, with the extra conditions that if it is in 0 then it cannot jump to the left and if it is in 3 then it cannot jump to the right. The resulting (X_t) has exactly the same dynamics as the barbershop chain of the exercise. Consider the discrete-time birth and death chain (Y_n) which makes a jump when there is an arrival of N_t. The transition matrix of (Y_n) on the state space $S = \{0,1,2,3\}$ is

$$\underline{\underline{P}} = \begin{pmatrix} 3/4 & 1/4 & 0 & 0 \\ 3/4 & 0 & 1/4 & 0 \\ 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 3/4 & 1/4 \end{pmatrix}$$

Note that $4\underline{\underline{P}} - \underline{\underline{I}} = \underline{\underline{G}}$, where $\underline{\underline{G}}$ is the infinitesimal generator matrix of (X_t), therefore the stationary distribution of the discrete chain (Y_n) is exactly the same as the stationary distribution of the continuous-time chain (X_t), see bottom of page 147 of scanned lecture notes. Now by the law of large numbers for the two-step chain (c.f. page 32 of scanned), we have $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathbb{1}[Y_k \le Y_{k+1}] = \pi(0)p(0,1) + \pi(1)p(1,2) + \pi(2)p(2,3) + \pi(3)p(3,3) = \frac{1}{4}$ (this is the fraction of jumps of (Y_n) corresponding to customer arrivals), moreover $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathbb{1}[Y_k = 3,\ Y_{k+1} = 3] = \pi(3)p(3,3) = \frac{1}{4}\pi(3)$ (this is the fraction of jumps of (Y_n) corresponding to customer arrivals when all chairs are occupied). Thus the fraction of arriving customers that find no empty chair is $\pi(3)$ and the fraction of arriving customers that do find an empty chair is $1 - \pi(3) = \frac{39}{40}$.

(b) Let $f(x) = E_x(T_0)$, i.e., the expected hitting time of state 0 if we start from state x. We want to find $f(3)$. We have $f(0) = 0$ and

$$f(1) = \frac{1}{4} + \frac{3}{4}0 + \frac{1}{4}f(2), \quad f(2) = \frac{1}{4} + \frac{3}{4}f(1) + \frac{1}{4}f(3), \quad f(3) = \frac{1}{3} + f(2).$$

Solving this system of equations we obtain $f(1) = \frac{13}{27}$, $f(2) = \frac{25}{27}$ and $f(3) = \frac{34}{27}$.

(c) $\sum_{x=0}^{3} \pi(x)\frac{1}{3}x = \frac{27}{40}\frac{0}{3} + \frac{9}{40}\frac{1}{3} + \frac{3}{40}\frac{2}{3} + \frac{1}{40}\frac{3}{3} = \frac{3}{20}$, because cutting the hair of x customers takes $\frac{1}{3}x$ time on average and $\pi(x)$ is the probability of finding x customers there if I enter the barbershop when it is in stationary state.