Stoch. Proc. HW assignment 12. Due Tuesday, December 12

Note: Each of the 3 questions is worth 10 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises with a horizontal line. Highlight the final answer. Paper format is preferred, but if you submit your homework electronically, pdf format is preferred.

1. Let us consider the discrete-time birth/death process (X_n) with state space $\mathbb{N}$ and transition rules

$$\begin{aligned} \mathbb{P}(\, X_{n+1} = x+1 \,|\, X_n = x \,) &= p_x, \quad x = 1,2,3,\dots \\ \mathbb{P}(\, X_{n+1} = x-1 \,|\, X_n = x \,) &= q_x, \quad x = 1,2,3,\dots, \\ \mathbb{P}(\, X_{n+1} = x \,|\, X_n = x \,) &= 1 - p_x - q_x, \quad x = 1,2,3,\dots \\ \mathbb{P}(\, X_{n+1} = 0 \,|\, X_n = 0 \,) &= 1. \end{aligned}$$

   Let us define the function $\alpha : \mathbb{N} \to \mathbb{R}$ by

$$\alpha(x) = \sum_{y=0}^{x-1} \frac{q_1 q_2 \dots q_y}{p_1 p_2 \dots p_y}. \qquad \text{(Note: } \alpha(0) = 0 \text{ and } \alpha(1) = 1)$$

   Show that (M_n) is a martingale, where $M_n := \alpha(X_n)$. *Hint:* See page 177 of scanned lecture notes.

2. You are in a casino. If you bet 1 dollar in the n'th round then your your net profit in that round is ξ_n dollars where $\mathbb{P}(\xi_n = +1) = p$, $\mathbb{P}(\xi_n = -1) = q$, $q+p = 1$, $p > 1/2$ and $\xi_1, \xi_2, \dots$ are i.i.d. In other words: with probability $q < 1/2$ you lose the bet and with probability $p = 1-q > 1/2$ you double your bet. You bet C_n dollars in the n'th round. Obviously, your betting strategy has to be *predictable*, i.e., when you make your n'th bet, you do not yet know the value of ξ_n, so let us assume that C_n is measurable w.r.t. $(\xi_1, \dots, \xi_{n-1})$. Denote by y_0 your initial wealth and by Y_n your total wealth after the end of the n'th round. You cannot go in debt, so let's assume $0 \le C_n \le Y_{n-1}$, $n > 0$. You are allowed to play N rounds.

   Your goal is to maximize your expected rate of return $\mathbb{E}(\log Y_N - \log y_0)$.

   (a) Define
$$f(x) := p \ln(1+x) + (1-p) \ln(1-x), \qquad 0 \le x \le 1.$$
   Show that f is strictly concave. Find $\max_{0 \le x \le 1} f(x)$ and show that $\max_{0 \le x \le 1} f(x) > 0$.

   (b) Show that for any predictable betting strategy we have $\mathbb{E}\big(\ln(Y_{n+1}) \,|\, (\xi_1, \dots, \xi_n)\big) = \ln(Y_n) + f(\frac{C_{n+1}}{Y_n})$.

   (c) Let $Z_n := \log Y_n - n\alpha$, where $\alpha = p \log p + q \log q + \log 2$. Prove that for any predictable betting strategy the process (Z_n) is a supermartingale. Show that this implies $\mathbb{E}(\log Y_N - \log y_0) \le N\alpha$.
   Hint: See page 182 of the scanned lecture notes for the definition of a *supermartingale.*

   (d) Describe the betting strategy for which Z_n is a martingale and that $\mathbb{E}(\log Y_N - \log y_0) = N\alpha$ is achieved by this strategy. (Sometimes they call this the *log-optimal portfolio* in economics)

3. There are two amoebae at time 0 in a Petri dish: one of them is red, the other one is blue. An amoeba splits after an EXP(1) distributed waiting time into two amoebae which are identical to the original one. Denote by X_n the number of blue amoebae after the n'th split. Note that $X_0 = 1$ and that the total number of amoebae in the Petri dish after the n'th split is $n+2$.

   (a) Calculate $\mathbb{P}(\, X_{n+1} = x \,|\, X_n = x \,)$ and $\mathbb{P}(\, X_{n+1} = x+1 \,|\, X_n = x \,)$ for $x \in \{1, \dots, n+1\}$.

   (b) Let us define $M_n = \frac{X_n}{n+2}$ (i.e., M_n is the fraction of blue amoebae). Show that (M_n) is a martingale.

   (c) Calculate $\mathbb{P}(\, X_2 = x \,)$ for $x = 1,2,3$. Calculate $\mathbb{P}(\, X_3 = x \,)$ for $x = 1,2,3,4$.

   (d) Based on the pattern that you see in the previous sub-exercise, make a conjecture about the distribution of X_n for general n and verify your conjecture using induction on n.