Stoch. Anal. HW assignment 1. Due Friday, September 15 at start of class

Note: Each of the 3 questions is worth 10 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises with a horizontal line. Highlight the final answer. If you submit your homework electronically, pdf format is preferred.

1. A heavy smoker tries to quit smoking from time to time. After a year of smoking, on New Year's Eve he makes the resolution of quitting smoking with probability 2/3. If he makes this resolution, he will be abstinent all year next year. If he decides not to make this resolution, he will continue smoking next year. After a year of abstinence, with probability 3/4 he becomes so proud of himself that on New Year's Eve he allows himself to smoke for a year; otherwise he continues being abstinent for a year. He continues with the same pattern each year. Since he also drinks heavily on New Year's Eve, his memory span becomes very limited, so he makes his decision about smoking in the upcoming year solely based on his smoking habits in the current year.

   If he smokes in 2017, what is the probability that he will smoke in 2027? Find a simple exact formula.

   Solution:

   Denote by μ_S^n the probability that he smokes in year $2017+n$ and denote by μ_A^n the probability that he is abstinent in year $2017+n$.

   We have $\mu_S^0 = 1$ and $\mu_A^0 = 0$.

   Observe that the recursion $\mu_S^{n+1} = \frac{1}{3}\mu_S^n + \frac{3}{4}\mu_A^n$ holds, moreover we have $\mu_S^n + \mu_A^n = 1$, thus

   $$\mu_S^{n+1} = \frac{1}{3}\mu_S^n + \frac{3}{4}(1-\mu_S^n) = \frac{3}{4} - \frac{5}{12}\mu_S^n. \tag{1}$$

   Let us first find the general solution of this first order linear recursion. The fixed point of the recursion is μ_S^* satisfying $\mu_S^* = \frac{3}{4} - \frac{5}{12}\mu_S^*$, thus $\mu_S^* = \frac{9}{17}$. The corresponding homogeneous linear recursion is $\mu_{hom}^{n+1} = -\frac{5}{12}\mu_{hom}^n$, the general solution of which is $\mu_{hom}^n = C\left(-\frac{5}{12}\right)^n$. Thus the general solution of the original inhomogeneous recursion is $\mu_S^n = \frac{9}{17} + C\left(-\frac{5}{12}\right)^n$. Now we have to choose $C = \frac{8}{17}$ in order to satisfy the initial condition $\mu_S^0 = 1$. Thus we have $\mu_S^n = \frac{9}{17} + \frac{8}{17}\left(-\frac{5}{12}\right)^n$.

   In particular, the probability that he will smoke in 2027 is $\mu_S^{10} = \frac{9}{17} + \frac{8}{17}\left(-\frac{5}{12}\right)^{10}$.

2. We consider a simple random walk on the graph $\mathbb{Z}$: in each step, the walker tosses a fair coin and makes one step to the left if the coin comes up heads and makes one step to the right if the coin comes up tails. The walker starts from the origin.

   (a) What is the probability that the walker reaches site -100 before reaching site 200?

   (b) What is the expected number of visits to site 50 before exiting the interval $[-99, 199]$?

   Solution:

   (a) Define $h(x)$ to be the probability that the walker reaches site -100 before reaching site 200 if it starts from x, where $x \in \mathbb{Z} \cap [-100, 200]$. We are looking for $h(0)$.

   We have $h(-100) = 1$ and $h(200) = 0$, these are the boundary conditions. For any $x \in \mathbb{Z} \cap [-99, 199]$, we have the recursion

   $$h(x) = \frac{1}{2}h(x-1) + \frac{1}{2}h(x+1). \tag{2}$$

   If $h(-99) = 1 + \Delta$, then by equation (2) we have $1 + \Delta = \frac{1}{2} + \frac{1}{2}h(-98)$, thus $h(-98) = 1 + 2\Delta$. Again by equation (2) we have $1 + 2\Delta = \frac{1}{2}(1+\Delta) + \frac{1}{2}h(-97)$, thus $1 + 3\Delta = h(-97)$. There is a pattern here. Let's try $h(x) = 1 + (x+100)\Delta$ and verify that (2) is indeed satisfied:

   $$\frac{1}{2}h(x-1) + \frac{1}{2}h(x+1) = \frac{1}{2}\left(1 + (x-1+100)\Delta\right) + \frac{1}{2}\left(1 + (x+1+100)\Delta\right) = 1 + (x+100)\Delta = h(x)$$

   Thus $0 = h(200) = 1 + (200+100)\Delta$, thus $\Delta = \frac{-1}{300}$. Therefore $h(x) = 1 - \frac{(x+100)}{300}$ for any $x \in \mathbb{Z} \cap [-100, 200]$. In words: equation (2) says that $h(x)$ is the arithmetic mean of $h(x-1)$ and $h(x+1)$, thus $h(-100), h(-99), \ldots, h(200)$ has to be an arithmetic sequence, i.e., on the interval $[-100, 200]$, the graph of h is a straight line that satisfies the boundary conditions.

   In particular, $h(0) = 1 + (0+100)\frac{-1}{300} = \frac{2}{3}$.

(b) Define $f(x)$ to be the expected number of visits to site 50 before the walker reaches either the site -100 or the site 200 if it starts from x, where $x \in \mathbb{Z} \cap [-100, 200]$. We are looking for $f(0)$.

We have $f(-100) = 0$ and $f(200) = 0$, these are the boundary conditions. For any $x \in \mathbb{Z} \cap [-99, 199]$, we have the recursion

$$f(x) = \frac{1}{2} f(x-1) + \frac{1}{2} f(x+1) + \mathbb{1}[x = 50]. \tag{3}$$

That is, for any $x \in (\mathbb{Z} \cap [-99, 199]) \setminus \{50\}$ the value of $f(x)$ is the arithmetic mean of its neighbours, just like in equation (2); and if $x = 50$ then we have

$$f(50) = \frac{1}{2} f(49) + \frac{1}{2} f(51) + 1. \tag{4}$$

We need to add 1 in (4) since we start with a visit to site 50 before we step to a neighbour.

We have learnt from the solution of (a) that the only unknown is $f(50)$, because on the interval $[-100, 50]$, the graph of f is a straight line and on the interval $[50, 200]$, the graph of f is again a (different) straight line. Thus $f(x) = \frac{f(50)}{150}(x + 100)$ if $x \in \mathbb{Z} \cap [-100, 50]$ and $f(x) = \frac{f(50)}{150}(200 - x)$ if $x \in \mathbb{Z} \cap [50, 200]$, so that the boundary conditions are satisfied. Plugging these formulas in (4) we obtain

$$f(50) = \frac{1}{2} \frac{f(50)}{150}(49 + 100) + \frac{1}{2} \frac{f(50)}{150}(200 - 51) + 1.$$

Solving this we obtain $f(50) = 150$, thus $f(0) = \frac{f(50)}{150}(x + 100) = 100$.

This is somewhat surprising at first sight: on average, the walker starting from site 0 visits site 50 a hundred times times before exiting the interval $[-99, 199]$.

3. Consider two dogs (a dachshund and a beagle) and three fleas. The fleas jump back and forth between the two dogs. From time to time, a uniformly chosen flea jumps from one dog to the other. Initially all of the fleas are on the dachshund. What is the expected total number of flea-jumps until the first time that all of the fleas are on the beagle?

Solution:

Denote by X_n the number of fleas on the beagle after the n'th flea-jump. We have $X_0 = 0$ since initially all of the fleas are on the dachshund. Fleas don't have memory and they all behave the same way, so (X_n) is a Markov chain with state space $S = \{0, 1, 2, 3\}$ and transition probabilities

$$\mathbb{P}[X_{n+1} = k + 1 \mid X_n = k] = \frac{3-k}{3}, \qquad \mathbb{P}[X_{n+1} = k - 1 \mid X_n = k] = \frac{k}{3}, \qquad k \in S.$$

In words: if there are k fleas on the beagle and $3 - k$ fleas on the dachshund, and if we pick one flea uniformly who jumps to the other dog, then the number of fleas on the beagle will increase by one with probability $\frac{3-k}{3}$ and decrease by one with probability $\frac{k}{3}$.

Denote by $f(x)$ the expected number of jumps until the first time that all of the 3 fleas are on the beagle if we start with x fleas on the beagle. We want to find $f(0)$. We have $f(3) = 0$ and

$$\begin{aligned}
f(0) &= 1 + f(1), \\
f(1) &= 1 + \frac{1}{3} f(0) + \frac{2}{3} f(2), \\
f(2) &= 1 + \frac{2}{3} f(1) + \frac{1}{3} f(3) = 1 + \frac{2}{3} f(1).
\end{aligned}$$

We added 1 to the r.h.s. of each equation because if you start from any $x \in S \setminus \{3\}$, then you start by making one jump, and then you find yourself in a situation where the game restarts from state $x \pm 1$.

Plugging in the first and third equation into the second equation, we obtain

$$f(1) = 1 + \frac{1}{3}(1 + f(1)) + \frac{2}{3}(1 + \frac{2}{3} f(1)),$$

solving this for $f(1)$ we obtain $f(1) = 9$, thus $f(0) = 10$.