Stoch. Anal. HW assignment 2. Due Friday, September 22 at start of class

Note: Each of the 3 questions is worth 10 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises with a horizontal line. Highlight the final answer. Paper format is preferred, but if you submit your homework electronically, pdf format is preferred.

1. Consider the Markov chain $(X_n)_{n=0}^{\infty}$ with state space $S = \{1, 2, 3\}$ and transition matrix

$$\underline{\underline{P}} = \begin{pmatrix} 1/6 & 2/3 & 1/6 \\ 1/6 & 1/6 & 2/3 \\ 1/2 & 1/6 & 1/3 \end{pmatrix}.$$

   Let us assume $X_0 = 1$ and define $T = \min\{\, n \;:\; X_n = 3 \,\}$ to be the hitting time of state 3.

   (a) Find $\mathbb{E}(T)$.

   (b) Find the vectors $\underline{v}, \underline{w}$ and the matrix $\underline{\underline{Q}}$ for which $\mathbb{P}[\, X_1 \neq 3, X_2 \neq 3, \dots, X_n \neq 3\,] = \underline{v}^T \underline{\underline{Q}}^n \underline{w}$ holds.
   Hint: For the idea of the solution, see page 18-19 of the scanned lecture notes.

   (c) Find an explicit formula for $\mathbb{P}[T = n]$ for any $n \in \mathbb{N}$.
   Hint: $\mathbb{P}[T = n] = \mathbb{P}[T > n-1] - \mathbb{P}[T > n]$ and use (b).

2. Suppose that an electronics store sells a certain type of gadget and uses the following inventory policy: if at the end of the day, the number of gadgets they have on hand is either one or zero then they order enough gadgets so that the total number of gadgets on hand at the beginning of the next day is 4. The gadgets ordered in the evening arrive next morning before the store opens. The demand for gadgets is D_n on the n'th day, and we assume that the random variables $(D_n)_{n=1}^{\infty}$ are i.i.d. with distribution

$$\mathbb{P}(D_n = 0) = \frac{1}{3}, \quad \mathbb{P}(D_n = 1) = \frac{1}{4}, \quad \mathbb{P}(D_n = 2) = \frac{1}{4}, \quad \mathbb{P}(D_n = 3) = \frac{1}{6}.$$

   Of course the store cannot sell more gadgets than they have on stock that day.

   Denote by X_n the number of gadgets the store has at hand at the end of the day n.

   (a) Briefly argue why (X_n) is a Markov chain and write down its transition matrix.

   (b) Briefly argue why this Markov chain is irreducible and find its stationary distribution (it is OK to use software for this calculation).

   (c) Suppose the store makes \$10 profit on each gadget sold but it costs \$2 to store one gadget overnight. Assuming that they have already been using the above inventory policy for a long time, roughly how much profit they will make next year?

3. Suppose you flip a fair coin repeatedly. The outcome of a coin flip is either „heads" or „tails".

   (a) What is the expected number of coin flips that you need to perform if you want to see two consecutive „heads"?

   (b) What is the expected number of coin flips that you need to perform if you want to see a „heads" followed by a „tails"?
   Hint: The underlying Markov chain is similar to the one discussed at the bottom of page 30 of the scanned lecture notes.