Stoch. Proc. HW assignment 2. Due Friday, September 22 at start of class

1. Consider the Markov chain $(X_n)_{n=0}^{\infty}$ with state space $S = \{1, 2, 3\}$ and transition matrix

$$\underline{\underline{P}} = \begin{pmatrix} 1/6 & 2/3 & 1/6 \\ 1/6 & 1/6 & 2/3 \\ 1/2 & 1/6 & 1/3 \end{pmatrix}.$$

Let us assume $X_0 = 1$ and define $T = \min\{\, n \,:\, X_n = 3\,\}$ to be the hitting time of state 3.

(a) Find $\mathbb{E}(T)$.

(b) Find the vectors $\underline{v}, \underline{w}$ and the matrix $\underline{\underline{Q}}$ for which $\mathbb{P}[\, X_1 \neq 3, X_2 \neq 3, \ldots, X_n \neq 3\,] = \underline{v}^T \underline{\underline{Q}}^n \underline{w}$ holds.

(c) Find an explicit formula for $\mathbb{P}[T = n]$ for any $n \in \mathbb{N}$.

Solution:

(a) Let $f(x)$ denote the expected value of the hitting time of state 3 if we start with $X_0 = x$, where $x = 1, 2, 3$. We want to find $f(1)$. We have $f(3) = 0$ (since if $X_0 = 3$ then $T = 0$) and

$$f(1) = 1 + \frac{1}{6} f(1) + \frac{2}{3} f(2) + \frac{1}{6} f(3), \qquad f(2) = 1 + \frac{1}{6} f(1) + \frac{1}{6} f(2) + \frac{2}{3} f(3). \tag{1}$$

Solving this system of linear equations gives $f(1) = \frac{18}{7}$ and $f(2) = \frac{12}{7}$.
In particular, if $X_0 = 1$ then $\mathbb{E}(T) = \frac{18}{7}$.

(b) We will argue that $\underline{v} = (1, 0)^T$, $\underline{w} = (1, 1)^T$ and $\underline{\underline{Q}} = \begin{pmatrix} 1/6 & 2/3 \\ 1/6 & 1/6 \end{pmatrix}$ (i.e., the matrix that we obtain from $\underline{\underline{P}}$ if we throw away the third row and third column). Let $\underline{\underline{P}} = (p(i,j))_{i,j=1}^3$. We have

$$\begin{aligned}
\mathbb{P}[T > n] &= \mathbb{P}[\, X_1 \neq 3, X_2 \neq 3, \ldots, X_n \neq 3\,] = \mathbb{P}[\, X_1 \in \{1,2\}, X_2 \in \{1,2\}, \ldots, X_n \in \{1,2\}\,] = \\
&\sum_{x_1 \in \{1,2\}} \sum_{x_2 \in \{1,2\}} \cdots \sum_{x_n \in \{1,2\}} \mathbb{P}[\, X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n\,] = \\
&\sum_{x_1 \in \{1,2\}} \sum_{x_2 \in \{1,2\}} \cdots \sum_{x_n \in \{1,2\}} p(1, x_1) p(x_1, x_2) \ldots p(x_{n-1}, x_n) = \\
&\sum_{x_1, \ldots x_{n-1} \in \{1,2\}} p(1, x_1) p(x_1, x_2) \ldots p(x_{n-1}, 1) + \sum_{x_1, \ldots x_{n-1} \in \{1,2\}} p(1, x_1) p(x_1, x_2) \ldots p(x_{n-1}, 2) \overset{(*)}{=} \\
&\underline{\underline{Q}}^n(1,1) + \underline{\underline{Q}}^n(1,2) = (1,0) \underline{\underline{Q}}^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \underline{v}^T \underline{\underline{Q}}^n \underline{w},
\end{aligned} \tag{2}$$

where the equation marked by $(*)$ follows from an argument analogous to the one on page 18-19 of the lecture notes.

(c) The eigenvalues of $\underline{\underline{Q}}$ are $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = -\frac{1}{6}$.

The corresponding right eigenvectors are $\underline{u}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\underline{u}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

We can express $\underline{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{3}{4} \underline{u}_1 + \frac{1}{4} \underline{u}_2$ in the eigenbasis. Therefore

$$\underline{\underline{Q}}^n \underline{w} = \frac{3}{4} \underline{\underline{Q}}^n \underline{u}_1 + \frac{1}{4} \underline{\underline{Q}}^n \underline{u}_2 = \frac{3}{4} \left(\frac{1}{2}\right)^n \underline{u}_1 + \frac{1}{4} \left(-\frac{1}{6}\right)^n \underline{u}_2, \tag{3}$$

and thus $\mathbb{P}[T > n] = \underline{v}^T \underline{\underline{Q}}^n \underline{w} = \frac{3}{4} \left(\frac{1}{2}\right)^n \cdot 2 + \frac{1}{4} \left(-\frac{1}{6}\right)^n \cdot (-2) = \frac{3}{2} \left(\frac{1}{2}\right)^n - \frac{1}{2} \left(-\frac{1}{6}\right)^n$.
(Note: $\mathbb{E}[T] = \sum_{n=0}^{\infty} \mathbb{P}[T > n] = \frac{18}{7}$ in accordance with the solution of part (a))
Finally: $\mathbb{P}[T = n] = \mathbb{P}[T > n-1] - \mathbb{P}[T > n] = \left(\frac{3}{2} \left(\frac{1}{2}\right)^{n-1} - \frac{1}{2} \left(-\frac{1}{6}\right)^{n-1}\right) - \left(\frac{3}{2} \left(\frac{1}{2}\right)^n - \frac{1}{2} \left(-\frac{1}{6}\right)^n\right)$
$\mathbb{P}[T = n] = \frac{3}{2} \left(\frac{1}{2}\right)^n + \frac{7}{2} \left(-\frac{1}{6}\right)^n$, $n = 1, 2, \ldots$

2. Suppose that an electronics store sells a certain type of gadget and uses the following inventory policy: if at the end of the day, the number of gadgets they have on hand is either one or zero then they order enough gadgets so that the total number of gadgets on hand at the beginning of the next day is 4. The gadgets ordered in the evening arrive next morning before the store opens. The demand for gadgets is D_n on the n'th day, and we assume that the random variables $(D_n)_{n=1}^{\infty}$ are i.i.d. with distribution

$$\mathbb{P}(D_n = 0) = \frac{1}{3}, \quad \mathbb{P}(D_n = 1) = \frac{1}{4}, \quad \mathbb{P}(D_n = 2) = \frac{1}{4}, \quad \mathbb{P}(D_n = 3) = \frac{1}{6}. \tag{4}$$

Of course the store cannot sell more gadgets than they have on stock that day.

Denote by X_n the number of gadgets the store has at hand at the end of the day n.

(a) Briefly argue why (X_n) is a Markov chain and write down its transition matrix.

(b) Briefly argue why this Markov chain is irreducible and find its stationary distribution (it is OK to use software for this calculation).

(c) Suppose the store makes \$10 profit on each gadget sold but it costs \$2 to store one gadget overnight. Assuming that they have already been using the above inventory policy for a long time, roughly how much profit they will make next year?

Solution:

(a) For $x \in \mathbb{R}$ let us denote by $x_+ := \max\{x, 0\}$ the positive part of x. We have

$$X_{n+1} = \varphi(X_n, D_{n+1}), \quad \text{where} \quad \varphi(x, d) := \begin{cases} (4-d)_+ & \text{if } x \in \{0,1\}, \\ (x-d)_+ & \text{if } x \in \{2,3,4\}. \end{cases} \tag{5}$$

Thus it is clear that X_{n+1} is a deterministic function of X_n and D_{n+1}, moreover D_{n+1} is independent from $(X_0, \ldots, X_n)$. In other words, given the present value of X_n, the past is irrelevant for predicting the future – the process (X_n) has the Markov property. The transition matrix on the state space $S = \{0, 1, 2, 3, 4\}$ is

$$\underline{\underline{P}} = \begin{pmatrix} 0 & 1/6 & 1/4 & 1/4 & 1/3 \\ 0 & 1/6 & 1/4 & 1/4 & 1/3 \\ 5/12 & 1/4 & 1/3 & 0 & 0 \\ 1/6 & 1/4 & 1/4 & 1/3 & 0 \\ 0 & 1/6 & 1/4 & 1/4 & 1/3 \end{pmatrix} \tag{6}$$

(b) It is easy to show that for any $x, y \in S$ we have either $p(x,y) > 0$ or $p^{(2)}(x,y) > 0$. For example $p(2,3) = 0$ but $p^{(2)}(2,3) \geq p(2,0)p(0,3) > 0$. Thus the Markov chain (X_n) is irreducible and therefore it has a unique stationary distribution $\underline{\pi} = (\pi_0, \pi_1, \ldots, \pi_4)$. In order to find it, we need to solve the system of linear equations

$$\begin{aligned} \pi_0 &= \pi_0 0 + \pi_1 0 + \pi_2 \frac{5}{12} + \pi_3 \frac{1}{6} + \pi_4 0, \\ \pi_1 &= \pi_0 \frac{1}{6} + \pi_1 \frac{1}{6} + \pi_2 \frac{1}{4} + \pi_3 \frac{1}{4} + \pi_4 \frac{1}{6}, \\ \pi_2 &= \pi_0 \frac{1}{4} + \pi_1 \frac{1}{4} + \pi_2 \frac{1}{3} + \pi_3 \frac{1}{4} + \pi_4 \frac{1}{4}, \\ \pi_3 &= \pi_0 \frac{1}{4} + \pi_1 \frac{1}{4} + \pi_2 0 + \pi_3 \frac{1}{3} + \pi_4 \frac{1}{4}, \\ \pi_4 &= \pi_0 \frac{1}{3} + \pi_1 \frac{1}{3} + \pi_2 0 + \pi_3 0 + \pi_4 \frac{1}{3}, \\ 1 &= \pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4. \end{aligned}$$

The first five equations are $\underline{\pi} = \underline{\pi}\,\underline{\underline{P}}$, while the sixth equation guarantees that $\underline{\pi}$ is a probability distribution. We have six equations and only five variables, but note that there is a redundancy in the first five equations, since if we add them up then we obtain the trivial identity

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 = \pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4. \tag{7}$$

(This identity follows from the fact that $\underline{\underline{P}}$ is a stochastic matrix.) We can thus throw away one of the first five equations (let's say the fifth one) and the set of solutions will remain the same. I used this Wolfram Mathematica command to solve the equations:

$$\text{LinearSolve}\big[\{\{-1,0,5/12,1/6,0\},\{1/6,-5/6,1/4,1/4,1/6\},\{1/4,1/4,-2/3,1/4,1/4\},\\ \{1/4,1/4,0,-2/3,1/4\},\{1,1,1,1,1\}\},\{0,0,0,0,1\}\big],$$

and the solution turned out to be

$$\pi_0 = \frac{71}{484}, \quad \pi_1 = \frac{299}{1452}, \quad \pi_2 = \frac{3}{11}, \quad \pi_3 = \frac{24}{121}, \quad \pi_4 = \frac{64}{363}. \tag{8}$$

(c) By the Law of Large Numbers for Markov Chains, the answer will be $356m$, where m is the stationary daily expected profit.

If they have 0 gadgets on the n'th evening then there is nothing to store overnight and thus the expected profit next day is $10 \cdot \mathbb{E}[D_{n+1}] = \frac{25}{2}$.

If they have 1 gadget on the n'th evening then they have to store it overnight and the expected profit next day is $10 \cdot \mathbb{E}[D_{n+1}] - 2 = \frac{21}{2}$.

If they have 2 gadgets on the n'th evening, then it might happen that the demand next day is bigger than the supply, so the expected profit next day is $10 \cdot \mathbb{E}[\max\{2, D_{n+1}\}] - 4 = \frac{41}{6}$.

If they have 3 gadgets on the n'th evening then the expected profit next day is $10 \cdot \mathbb{E}[D_{n+1}] - 6 = \frac{13}{2}$.

If they have 4 gadgets on the n'th evening then the expected profit next day is $10 \cdot \mathbb{E}[D_{n+1}] - 8 = \frac{9}{2}$.

Therefore

$$m = \pi_0 \frac{25}{2} + \pi_1 \frac{21}{2} + \pi_2 \frac{41}{6} + \pi_3 \frac{13}{2} + \pi_4 \frac{9}{2} = \frac{961}{121} \tag{9}$$

Thus the profit that the store makes on gadgets next year is roughly $365m \approx 2898.88$ dollars.

Remark: This calculation in (c) was a bit non-rigorous. We said that we applied the law of large numbers for Markov chains (page 28 of the scanned lecture notes) for the Markov chain (X_n), but in fact we applied it to a Markov chain with bigger state space, as we now explain.

If we define $\widetilde{X}_n = (X_n, D_{n+1})$ then one can see that $(\widetilde{X}_n)$ is also a Markov chain with state space $\widetilde{S} = \{0,1,2,3,4\} \times \{0,1,2,3\}$ and transition rule

$$\mathbb{P}\left[\widetilde{X}_{n+1} = (x', d') \,|\, \widetilde{X}_n = (x, d)\right] = \mathbb{1}\left[\, x' = \varphi(x,d)\,\right] \mu(d'), \text{ where}$$
$$\mu(0) = \frac{1}{3},\ \mu(1) = \frac{1}{4},\ \mu(2) = \frac{1}{4},\ \mu(3) = \frac{1}{6}, \tag{10}$$

and φ was defined in equation (5) and μ is the distribution of D_{n+2}, see equation (4). One can see that $(\widetilde{X}_n)$ is also an irreducible Markov chain with stationary distribution

$$\widetilde{\pi}(x, d) = \pi_x \mu(d),$$

where π_x appears in (8) and $\mu(d)$ is defined in (10).

Now m (the stationary expected daily profit) can be written as

$$m = \sum_{\widetilde{x} \in \widetilde{S}} f(\widetilde{x}) \widetilde{\pi}(\widetilde{x}), \tag{11}$$

where \$$f(\widetilde{x})$ is the profit made on day $n+1$ if $\widetilde{X}_n = \widetilde{x} = (x, d)$, i.e., if $X_n = x$ and $D_{n+1} = d$:

$$f(\widetilde{x}) = f(x, d) := 10 \cdot \max\{\psi(x), d\} - 2x, \quad \text{where} \quad \psi(x) := \begin{cases} 4 & \text{if } x \in \{0,1\}, \\ x & \text{if } x \in \{2,3,4\}. \end{cases}$$

The formula (11) gives the same value of m as in (9).

Thus we can indeed apply the law of large numbers for Markov chains to $(\widetilde{X}_n)$ to infer that the yearly profit is roughly $356m$.

3. Suppose you flip a fair coin repeatedly. The outcome of a coin flip is either „heads” or „tails”.

   (a) What is the expected number of coin flips that you need to perform if you want to see two consecutive „heads”?

   (b) What is the expected number of coin flips that you need to perform if you want to see a „heads” followed by a „tails”?

   Hint: The underlying Markov chain is similar to the one discussed at the bottom of page 30 of the scanned lecture notes.

Solution:

(a) Let η_k denote the result of the k'th coin flip ($k = 1, 2, 3, \dots$), i.e., $\eta_k = H$ if the result is heads and $\eta_k = T$ if the result is tails. Denote by

$$X_k = (\eta_{k+1}, \eta_{k+2}), \quad \text{where} \quad k = 0, 1, 2, \dots$$

Then (X_k) is a Markov chain with state space $\{HH, HT, TH, TT\}$ and transition matrix

$$\begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}$$

Let us define $f(x)$ as the expected number of steps until the Markov chain reaches state HH if the initial state is x, where $x \in \{HH, HT, TH, TT\}$. We have $f(HH) = 0$ and

$$\begin{aligned} f(HT) &= 1 + \frac{1}{2} f(TH) + \frac{1}{2} f(TT), \\ f(TH) &= 1 + \frac{1}{2} f(HH) + \frac{1}{2} f(HT), \\ f(TT) &= 1 + \frac{1}{2} f(TH) + \frac{1}{2} f(TT). \end{aligned}$$

Solving this we obtain $f(TH) = 4$, $f(TT) = 6$ and $f(HT) = 6$.

Denote by m the expected number of coin flips that you need to perform if you want to see two consecutive „heads”. Now

$$m = 2 + \frac{1}{4} \left(f(HH) + f(HT) + f(TH) + f(TT) \right)$$

because we have to throw at least twice and then we find ourselves in state X_0 of our Markov chain which is uniformly distributed on the state space. Therefore $m = 6$.

(b) The expected number of coin flips that you need to perform until you see the first heads is 2 (expected value of a $\text{GEO}(\frac{1}{2})$ random variable). After this, the expected number of further flips that you need to perform until the first tails is also 2. Thus the expected number of coin flips that you need to perform if you want to see a „heads” followed by a „tails” is equal to 4.