Stoch. Proc. HW assignment 3. Due Friday, September 29 at start of class

Note: Each of the 3 questions is worth 10 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises with a horizontal line. Highlight the final answer. Paper format is preferred, but if you submit your homework electronically, pdf format is preferred.

1. The USA and Canada both have 20 museums of modern art. Each of these 40 museums have one abstract expressionist painting, namely 20 Pollock paintings and 20 Rothko paintings. Each month, the international art committee chooses uniformly at random an abstract expressionist painting in Canada and swaps it with a uniformly chosen abstract expressionist painting in the US in order to entertain the art-loving audience in both countries with something new. Denote by X_n the number of Pollock paintings in the US in the n'th month.
   (a) Show that (X_n) is a birth/death chain and find its transition probabilities.
   (b) Find the stationary distribution of (X_n).
   Hint: The nicest way to solve (b) is to define a Markov chain with a much bigger state space (somewhat similarly to the case of the Ehrenfest chain), find the stationary distribution of the big Markov chain and then project the result back to the original small Markov chain.

2. Miles Raymond is a wine enthusiast, he drinks one bottle every day and puts the empty bottle on the shelf. His wife Maya, each evening, with probability 1/2 looks at the shelf and recycles all of the wine bottles. Every time five wine bottles pile up on the shelf, Miles himself immediately recycles all of them. Let B_n denote the number of wine bottles on the self on the n'th day (after the potential recycling).
   (a) Write down the transition matrix of the Markov chain (B_n).
   (b) Show that this Markov chain is irreducible and find its stationary distribution.
   (c) Suppose they have been doing this protocol for a long long time. What is the expected number of wine bottles on the shelf tonight (after the potential recycling)?

3. There are N locations where a flower can grow. There are two types of flowers (dandelions and chamomiles, say) and each location is occupied by exactly one flower. Next year a new flower grows on each location and the seed of that flower came from a uniformly chosen flower from last year's population (this choice is independent for all locations). Denote by X_n the total number of dandelions in the n'th year.
   (a) Write down the transition matrix of the Markov chain X_n.
   Hint: It is a good idea to first recall the notion of *binomial distribution*.
   (b) If $N = 5$, assuming we start with $X_0 = k$ dandelions, calculate the probability that we end up with all-dandelion population on the long run.
   Hint: Use symmetry to reduce the number of unknowns.
   (c) Now $N = 1000$. Assuming we start with $X_0 = k$ dandelions, calculate the probability that we end up with all-dandelion population on the long run.
   Hint: In order to solve (c), first solve (b) and make a guess about the form of the solution for the case of $N = 1000$, and then verify that your guess was indeed correct.