Stoch. Proc. HW assignment 4. Due Friday, October 6 at start of class

1. Each day it rains with probability 1/4, independently from what happens on other days. Moreover if it rains then it rains exactly at noon. An old gardener waters his garden in the afternoon if he sees that the garden has not been watered (by either rain or himself) in the last three days (i.e., today, yesterday and the day before yesterday). Roughly how many times does he water his garden per year?

Solution:

The garden can be in four different states in the afternoon (after potential rain, but before potential watering by the gardener): state 0 means that it rained today at noon, state 1 means that the last time the garden was watered (by rain or gardener) was yesterday, state 2 means that the last time the garden was watered (by rain or gardener) was the day before yesterday and state 3 means that the last time the garden was watered (by rain or gardener) was three days ago. The state space is $S = \{0, 1, 2, 3\}$. Transition matrix:

$$\underline{\underline{P}} = \begin{pmatrix} 1/4 & 3/4 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 \\ 1/4 & 0 & 0 & 3/4 \\ 1/4 & 3/4 & 0 & 0 \end{pmatrix}$$

The stationary distribution $\pi = (\pi_0, \pi_1, \pi_2, \pi_3)$ solves the system of equations

$$\begin{aligned} \pi_0 &= \frac{1}{4}\pi_0 + \frac{1}{4}\pi_1 + \frac{1}{4}\pi_2 + \frac{1}{4}\pi_3, \\ \pi_1 &= \frac{3}{4}\pi_0 + \frac{3}{4}\pi_3, \\ \pi_2 &= \frac{3}{4}\pi_1, \\ \pi_3 &= \frac{3}{4}\pi_2, \\ 1 &= \pi_0 + \pi_1 + \pi_2 + \pi_3. \end{aligned}$$

The first four equations state that π is a left eigenvector of $\underline{\underline{P}}$ corresponding to the eigenvalue 1. The fifth equation guarantees that π is a probability distribution. The sum of the first four equations is the trivial identity

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 = \pi_0 + \pi_1 + \pi_2 + \pi_3,$$

this trivial identity follows from the property that the sum of each row of $\underline{\underline{P}}$ is equal to 1. Therefore the first four equations together are redundant, we can throw away the first equation.

Solving the resulting system of linear equations we obtain

$$\pi_0 = \frac{1}{4}, \quad \pi_1 = \frac{48}{148}, \quad \pi_2 = \frac{36}{148}, \quad \pi_3 = \frac{27}{148}.$$

The gardener waters the garden if it is in state 3. By the law of large numbers for Markov chains, he waters the garden roughly $365 \cdot \pi_3$ times per year, that is, roughly 66.58 times per year.

Alternative solution:

Let us consider the renewal process (see page 73 of the scanned lecture notes) where the watering (by either rain or gardener) plays the role of a renewal. If a renewal occurred today then the expected time (in days) that passes until the next renewal is

$$m = \frac{1}{4} \cdot 1 + \frac{3}{4}\frac{1}{4} \cdot 2 + \left(\frac{3}{4}\right)^2 \frac{1}{4} \cdot 3 + \left(\frac{3}{4}\right)^3 \cdot 3 = \frac{37}{16}.$$

Thus by the classical law of large numbers for i.i.d. random variables, the number of renewals per year is roughly $365/m \approx 157.83$. A renewal period ends with the gardener watering the garden with probability $\left(\frac{3}{4}\right)^3$. Thus, again by the classical law of large numbers, the number of renewal periods in a year that end with watering is roughly equal to $157.83 \cdot \left(\frac{3}{4}\right)^3 \approx 66.58$.

2. There is a queue of university students standing in line in front of the Central Office of Education. Each second, a new student arrives and joins the queue with probability p and a student is served in the office and thus leaves the queue with probability q. The maximal length of the queue is 30. Any student arriving when there are already 30 students standing in line will be immediately sent away.

   (a) If $p = 0.01$ and $q = 0.012$ and assuming that the queue is stationary, what is the probability that the length of the queue is at most 15?

   (b) If $p = 0.01$ and $q = 0.008$ and assuming that the queue is stationary, what is the probability that the length of the queue is at most 15?

   (c) If $p = 0.01$ and $q = 0.01$ and assuming that the queue is stationary, what is the probability that the length of the queue is at most 15?

   (d) If $p = 0.01$ and $q = 0.012$ and the current queue length is 15, what is the probability that the queue becomes empty before it reaches its maximal length?

   (e) If $p = 0.01$ and $q = 0.008$ and the current queue length is 15, what is the probability that the queue becomes empty before it reaches its maximal length?

   (f) If $p = 0.01$ and $q = 0.01$ and the current queue length is 15, what is the probability that the queue becomes empty before it reaches its maximal length?

Solution: Denote by X_n the length of the queue at time n (time is measured in seconds). Then (X_n) is birth-and-death process with transition probabilities $p_x = p$ for any $0 \le x \le 29$ and $q_x = q$ for any $1 \le x \le 30$ (here we cheated since we ignored the possibility that a student arrives and another one departs in the same second, which happens with probability pq, which is small compared to p and q). Also note that $\mathbb{P}(X_{n+1} = X_n)$ is close to 1, however this fact will not manifest itself in our calculations.

Now we recall some facts about birth-and-death processes from page 45 of the scanned lecture notes.

Stationary distribution:

$$\pi(x) = \frac{p_0 p_1 \cdots p_{x-1}}{q_1 q_2 \cdots q_x} \pi_0 = \left(\frac{p}{q}\right)^x \pi_0, \quad \pi_0 = \left(\sum_{x=0}^{30} \left(\frac{p}{q}\right)^x\right)^{-1}.$$

Let $h(x)$ be the probability of the event that the length of the queue reaches 30 before reaching 0.

$$h(15) = \frac{\alpha_{15}}{\alpha_{30}}, \qquad \alpha_x = \sum_{y=0}^{x-1} \left(\frac{q}{p}\right)^y.$$

(a) $\frac{p}{q} = \frac{5}{6}$ and thus $\frac{1}{\pi_0} = \sum_{x=0}^{30} \left(\frac{5}{6}\right)^x \approx \sum_{x=0}^{\infty} \left(\frac{5}{6}\right)^x = \frac{1}{1-\frac{5}{6}} = 6$, thus $\pi_x \approx \frac{1}{6}\left(\frac{5}{6}\right)^x$. The probability that the length of the queue is at most 15 is very high: $\sum_{x=0}^{15} \pi_x \approx 1 - \left(\frac{5}{6}\right)^{15} \approx 0.935$

(b) $\frac{p}{q} = \frac{5}{4}$ and thus $\frac{1}{\pi_0} = \sum_{x=0}^{30} \left(\frac{5}{4}\right)^x \approx \frac{\left(\frac{5}{4}\right)^{31}-1}{\frac{5}{4}-1} \approx 4\left(\frac{5}{4}\right)^{31} = 5\left(\frac{5}{4}\right)^{30}$, thus $\pi_{30-x} \approx \frac{1}{5}\left(\frac{4}{5}\right)^x$. The probability that the length of the queue is at most 15 is very small: $\sum_{x=0}^{15} \pi_x = 1 - \sum_{x=0}^{14} \pi_{30-x} \approx \left(\frac{4}{5}\right)^{14} \approx 0.04$

(c) $\frac{p}{q} = 1$, $\pi_x = \pi_0$, the stationary distribution is uniform, thus the probability that the length of the queue is at most 15 is $\frac{16}{31} \approx \frac{1}{2}$.

(d) $\frac{q}{p} = \frac{6}{5}$, $\alpha_{15} = \sum_{y=0}^{14} \left(\frac{6}{5}\right)^y = \frac{\left(\frac{6}{5}\right)^{15}-1}{\frac{6}{5}-1}$, thus $h(15) = \frac{\left(\frac{6}{5}\right)^{15}-1}{\left(\frac{6}{5}\right)^{30}-1} \approx \left(\frac{5}{6}\right)^{15} \approx 0.065$. The probability that the queue becomes empty before it reaches its maximal length is very high: $1 - 0.065 = 0.935$.

(e) $\frac{q}{p} = \frac{4}{5}$, $\alpha_{15} = \sum_{y=0}^{14} \left(\frac{4}{5}\right)^y = \frac{1-\left(\frac{4}{5}\right)^{15}}{1-\frac{4}{5}}$, thus $h(15) = \frac{1-\left(\frac{4}{5}\right)^{15}}{1-\left(\frac{4}{5}\right)^{30}} \approx 1 - \left(\frac{4}{5}\right)^{15}$. The probability that the queue becomes empty before it reaches its maximal length is small: $\left(\frac{4}{5}\right)^{15} \approx 0.035$

(f) $\frac{q}{p} = 1$, $\alpha_x = x$, $h(15) = \frac{1}{2}$. The probability that the queue becomes empty before it reaches its maximal length is $1 - h(15) = \frac{1}{2}$.

Moral of the story: Arrival rate: p, service rate: q. If $p < q$ then the queue is short, if $p > q$ then the queue is long, if $p = q$ then the queue length is all over the place.

3. Consider the Markov chain on state space $\{1,2,3,4,5\}$ with transition matrix

$$\underline{\underline{P}} = \begin{pmatrix} 0 & 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & 3/4 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1}$$

(a) Show that the Markov chain is irreducible by drawing its accessibility graph.
(b) What is the period of this Markov chain?
(c) Find the stationary distribution $(\pi_1, \dots, \pi_5)$ of the Markov chain.
(d) Find $\underline{\underline{P}}^{2018}$.

Solution:

(a) Sorry, I won't draw it.
(b) The period is equal to 3.
(c) First note that the walker returns to state 1 in every third step. Thus it spends one third of the time in state 1 on the long run:

$$\pi_1 = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{1}[X_k = 1] = \frac{1}{3}.$$

From the second and third column of the transition matrix we see that

$$\pi_2 = \frac{1}{3}\pi_1 = \frac{1}{9}, \qquad \pi_3 = \frac{2}{3}\pi_1 = \frac{2}{9}.$$

From the fourth and fifth column of the transition matrix we obtain

$$\pi_4 = \frac{1}{4}\pi_2 + \frac{1}{2}\pi_3 = \frac{5}{36}, \qquad \pi_5 = \frac{3}{4}\pi_2 + \frac{1}{2}\pi_3 = \frac{7}{36}.$$

Sanity check: indeed $\pi_1 = \pi_4 + \pi_5$, as required by the first column of the transition matrix.
Sanity check: indeed $\pi_1 + \dots + \pi_5 = 1$.

(d) The first row of $\underline{\underline{P}}^{2018}$ is the distribution of X_{2018} we start from state $X_0 = 1$. Note that if $X_0 = 1$ then $\mathbb{P}[X_{2016} = 1] = 1$ (since 2016 is divisible by 3) and thus $\mathbb{P}[X_{2017} = 2] = \frac{1}{3}$ and $\mathbb{P}[X_{2017} = 3] = \frac{2}{3}$, moreover

$$\mathbb{P}[X_{2018} = 4] = \mathbb{P}[X_{2017} = 2]p(2,4) + \mathbb{P}[X_{2017} = 3]p(3,4) = \frac{1}{3}\frac{1}{4} + \frac{2}{3}\frac{1}{2} = \frac{5}{12},$$
$$\mathbb{P}[X_{2018} = 5] = \mathbb{P}[X_{2017} = 2]p(2,5) + \mathbb{P}[X_{2017} = 3]p(3,5) = \frac{1}{3}\frac{3}{4} + \frac{2}{3}\frac{1}{2} = \frac{7}{12}.$$

Therefore the first row of $\underline{\underline{P}}^{2018}$ is $(0, 0, 0, \frac{5}{12}, \frac{7}{12})$.
The second row of $\underline{\underline{P}}^{2018}$ is the distribution of X_{2018} if we start from state $X_0 = 2$. Note that in this case $\mathbb{P}[X_2 = 1] = 1$ and thus $\mathbb{P}[X_{2018} = 1] = 1$, thus the second row of $\underline{\underline{P}}^{2018}$ is $(1, 0, 0, 0, 0)$. Similarly we can calculate all rows:

$$\underline{\underline{P}}^{2018} = \begin{pmatrix} 0 & 0 & 0 & 5/12 & 7/12 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 2/3 & 0 & 0 \\ 0 & 1/3 & 2/3 & 0 & 0 \end{pmatrix}.$$

Remark: Clearly the sequence $\underline{\underline{P}}^n$ does not converge as $n \to \infty$ as the Markov chain is periodic. However let us note that $\underline{\underline{P}}^{n+3} = \underline{\underline{P}}^n$ if $n \geq 3$ and thus if we define $\underline{\underline{Q}} = \frac{1}{3}\left(\underline{\underline{P}}^3 + \underline{\underline{P}}^4 + \underline{\underline{P}}^5\right)$ then $\underline{\underline{Q}}\,\underline{\underline{P}} = \underline{\underline{Q}}$, and since the stationary distribution is unique this must mean that every row of $\underline{\underline{Q}}$ is the stationary distribution. As a consequence, $\lim_{n\to\infty} \frac{1}{3}\left(\underline{\underline{P}}^n + \underline{\underline{P}}^{n+1} + \underline{\underline{P}}^{n+2}\right) = \underline{\underline{Q}}$, meaning that if we „randomize" the time parameter in a way that the remainder modulo the period is uniformly distributed then convergence to stationarity does hold. This final conclusion also holds for any periodic irreducible Markov chain.