Stoch. Proc. HW assignment 5. Due Friday, October 13 at start of class

Note: Each of the 3 questions is worth 10 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises with a horizontal line. Highlight the final answer. Paper format is preferred, but if you submit your homework electronically, pdf format is preferred.

1. Let us consider the Markov chain with state space $S = \{1, 2, 3, 4, 5\}$ and transition matrix

$$\underline{\underline{P}} = \begin{pmatrix} 1/3 & 0 & 1/3 & 1/3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1/2 & 1/4 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

   (a) Partition S into irreducible components and find out which state is recurrent and which one is transient. *Hint:* Use the definitions and results stated on pages 64-67 of the scanned lecture notes.

   (b) Characterize the family of stationary distributions of this Markov chain. Check by hand that the distributions you have found are indeed stationary. *Hint:* Use the theorem stated on page 69 of the scanned lecture notes.

   (c) Starting from state 1, what is the probability of reaching state 2 before state 5?

   (d) Find $\lim_{n\to\infty} \underline{\underline{P}}^n$. *Hint:* Use the ideas from parts (b) and (c).

2. Let $\underline{\underline{P}} = (p(x, y))_{x,y\in S}$ denote the transition matrix of an irreducible Markov chain (X_n) with finite state space S. Show that the following three statements are equivalent:

   (a) The Markov chain (X_n) is reversible.

   (b) There exists a function $\varphi : S \to (0, +\infty)$ satisfying $\varphi(x)p(x, y) = \varphi(y)p(y, x)$ for any $x, y \in S$.

   (c) For any $k \in \mathbb{N}$ and any $x_1, x_2, \ldots, x_k \in S$ we have

$$p(x_1, x_2)p(x_2, x_3)\ldots p(x_{k-1}, x_k)p(x_k, x_1) = p(x_1, x_k)p(x_k, x_{k-1})\ldots p(x_3, x_2)p(x_2, x_1).$$

   Hint: Try proving $(a) \implies (c) \implies (b) \implies (a)$. In order to show that $(c) \implies (b)$, argue like this: you have to construct φ satisfying (b). Let us fix some $x^* \in S$. Without loss of generality, you may assume $\varphi(x^*) = 1$. Now it is quite easy to guess the φ value of the neighbours of x^* in the accessibility graph, then the φ value of the neighbours of the neighbours, etc. Use (c) to make sure that φ is unambiguously defined and that it indeed satisfies (b) for any $x, y \in S$.

3. Recall the notion of a *renewal process* from page 73 of the lecture notes. Let $\tau_1, \tau_2, \ldots$ denote i.i.d. random variables which have the same distribution as the random variable τ. We assume that τ takes positive integer values and also assume that $\mathbb{P}(\tau \le M) = 1$ for some $M \in \mathbb{N}$. Now τ_k is the length of the k'th renewal interval. Denote by $p_x = \mathbb{P}[\tau = x]$ for any $x \in \{1, \ldots, M\}$. Let us define $T_n = \sum_{i=1}^{n} \tau_i$, thus T_n is the time of the n'th renewal. Denote by α_t the time that has elapsed since the last renewal at time t and denote by β_t the time until the next renewal at time t, where $t = 0, 1, 2, \ldots$ (if $t = T_n$ is a renewal time then we let $\alpha_t = 0$ and $\beta_t = \tau_{n+1}$). Denote by $\gamma_t = \alpha_t + \beta_t$ the total length of the renewal interval that contains t.

   (a) Briefly argue that $Y_t := (\alpha_t, \beta_t)$, $t = 0, 1, 2, \ldots$ is an irreducible Markov chain on the state space

$$S = \{\, (x, y) \,:\, x \in \{0, \ldots, M-1\},\ y \in \{1, \ldots, M\},\ p_{x+y} > 0 \,\}$$

   and describe the transition rules of this Markov chain.

   (b) What is the period of this Markov chain?

   (c) Find the stationary distribution π of this Markov chain.
   Hint: Denote by $\alpha_\infty, \beta_\infty, \gamma_\infty$ the time that has elapsed since the last renewal, time until the next renewal and the total length of the current renewal interval in the stationary state. Show that

$$\pi(\alpha_\infty = x, \beta_\infty = y) = \frac{p_{x+y}}{\mathbb{E}(\tau)}.$$

   (d) Deduce from the result of the previous sub-exercise that $\mathbb{P}(\gamma_\infty = z) = \frac{z p_z}{\mathbb{E}(\tau)}$ for any $z \in \{1, \ldots, M\}$.
   Remark: Isn't it strange that the distribution of the total length of the renewal interval that contains a point t in time in the very distant future is different from the distribution of a renewal interval? This fact is often referred to as the *waiting time paradox*.