Stoch. Proc. HW assignment 6. Due Friday, October 20 at start of class

Note: Each of the 3 questions is worth 10 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises with a horizontal line. Highlight the final answer. Paper format is preferred, but if you submit your homework electronically, pdf format is preferred.

1. Recall the notion of a *renewal process* from page 73 of the lecture notes. Let $\tau_1, \tau_2, \dots$ denote i.i.d. random variables which have the same distribution as the random variable τ. We assume that τ takes positive integer values. Now τ_k is the length of the k'th renewal interval. Denote by $p_x = \mathbb{P}[\tau = x]$ for any $x \in \mathbb{N}$. Let us define $T_n = \sum_{i=1}^n \tau_i$, thus T_n is the time of the n'th renewal. Denote by α_t the time that has elapsed since the last renewal at time t, where $t = 0, 1, 2, \dots$ (if $t = T_n$ is a renewal time then we let $\alpha_t = 0$).

   (a) Briefly argue that α_t, $t = 0, 1, 2, \dots$ is an irreducible Markov chain and write down the state space and the transition matrix of this Markov chain.

   (b) What do we have to assume about the distribution of τ if we want the Markov chain (α_t) to be positive recurrent? In the positive recurrent case, find the stationary distribution of the Markov chain (α_t).

   (c) Verify that the earlier homework exercise pertaining to the wine bottle recycling habits of Miles Raymond is a special case of this exercise. Find the distribution of τ corresponding to the wine bottle exercise and use part (b) of this exercise to re-derive the stationary distribution in the case of the wine bottle exercise.

2. A hunter is trying to shoot a rabbit, the rabbit tries to escape. The rabbit is running around on the locations indexed by $S = \{-10, -9, -8, \dots, 8, 9, 10\}$. The movement of the rabbit is unpredictable, she performs simple, symmetric random walk on S. In each time-step, the rabbit moves to a location adjacent to its current location, and in each time-step, the hunter shoots at the rabbit (first the hunter shoots, then the rabbit moves). Each shot of the hunter is successful with probability $1/100$, but he misses the rabbit with probability $99/100$. As soon as the rabbit reaches the rabbit-holes located at site -10 and site 10, she is safe. If the rabbit initially starts from site 0, what is the probability that she survives this encounter with the hunter?

   Hint: Use the method of homogeneous linear difference equations that we have already used to solve the gambler's ruin problem, see page 21-22-23 of the scanned lecture notes.

3. Let $X_1, X_2, \dots$ denote i.i.d. non-negative integer-valued random variables which have the same distribution as the random variable X.

   Let $p_k = \mathbb{P}[X = k], k = 0, 1, 2, \dots$. Let us assume that $p_0 > 0$ and that $\sum_{k=0}^\infty k p_k = \mathbb{E}(X) > 1$.

   Let $Y_0 = 0$ and let us recursively define

$$Y_n = \max\{\, 0, Y_{n-1} + 1 - X_n \,\}, \qquad n = 1, 2, 3, \dots$$

   (a) Write down the transition matrix of the irreducible Markov chain (Y_n).

   (b) Let $G(z) := \sum_{k=0}^\infty z^k p_k$. Show that the equation $z = G(z)$ has exactly one solution $z^* \in (0, 1)$. *Hint:* $G(0) = ?$, $G(1) = ?$, $G'(1) = ?$, what can you say about $G''(z)$ if $0 \le z \le 1$?

   (c) Find the stationary distribution of the Markov chain (Y_n).

   Hint: We have already found the stationary distribution in the special case when

$$\mathbb{P}[X = 2] = q, \qquad \mathbb{P}[X = 0] = p, \qquad \mathbb{P}[X = 1] = 1 - p - q,$$

   see page 81-82 of the scanned lecture notes. The stationary distribution in the general case will look very similar to the stationary distribution in the special case, but note that the method of proof has to be different, because in the general case (Y_n) is not necessarily a birth-and-death chain!