Stoch. Proc. HW assignment 6. Due Friday, October 20 at start of class

1. Recall the notion of a *renewal process* from page 73 of the lecture notes. Let $\tau_1, \tau_2, \dots$ denote i.i.d. random variables which have the same distribution as the random variable τ. We assume that τ takes positive integer values. Now τ_k is the length of the k'th renewal interval. Denote by $p_x = \mathbb{P}[\tau = x]$ for any $x \in \mathbb{N}$. Let us define $T_n = \sum_{i=1}^n \tau_i$, thus T_n is the time of the n'th renewal. Denote by α_t the time that has elapsed since the last renewal at time t, where $t = 0, 1, 2, \dots$ (if $t = T_n$ is a renewal time then we let $\alpha_t = 0$).

   (a) Briefly argue that α_t, $t = 0, 1, 2, \dots$ is an irreducible Markov chain and write down the state space and the transition matrix of this Markov chain.

   (b) What do we have to assume about the distribution of τ if we want the Markov chain (α_t) to be positive recurrent? In the positive recurrent case, find the stationary distribution of the Markov chain (α_t).

   (c) Verify that the earlier homework exercise pertaining to the wine bottle recycling habits of Miles Raymond is a special case of this exercise. Find the distribution of τ corresponding to the wine bottle exercise and use part (b) of this exercise to re-derive the stationary distribution in the case of the wine bottle exercise.

Solution:

(a) The state space is $\{0, 1, 2, \dots, M-1\}$, where $M = \sup\{\, x \,:\, p_x > 0 \,\}$. Of course, $M = \infty$ is possible in which case the state space is $\mathbb{N}$. Let us define

$$a_x = \mathbb{P}[\tau \geq x] = \sum_{y \geq x} p_y.$$

Note that $a_1 = 1$, $a_M = p_M$ and $p_x + a_{x+1} = a_x$.
The transition matrix on the state space $\{0, 1, 2, \dots, M-1\}$ is

$$\underline{\underline{P}} = \begin{pmatrix} p_1/a_1 & a_2/a_1 & 0 & 0 & 0 & \dots & 0 \\ p_2/a_2 & 0 & a_3/a_2 & 0 & 0 & \dots & 0 \\ p_3/a_3 & 0 & 0 & a_4/a_3 & 0 & \dots & 0 \\ p_4/a_4 & 0 & 0 & 0 & a_5/a_4 & \dots & 0 \\ \vdots & 0 & 0 & 0 & 0 & \ddots & 0 \\ p_{M-1}/a_{M-1} & 0 & 0 & 0 & \dots & 0 & a_M/a_{M-1} \\ p_M/a_M & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

because $\mathbb{P}\,(\, \alpha_{t+1} = x+1 \,|\, \alpha_t = x \,) = \mathbb{P}[\tau > x+1 \,|\, \tau > x] = a_{x+2}/a_{x+1}$ and
$\mathbb{P}\,(\, \alpha_{t+1} = 0 \,|\, \alpha_t = x \,) = \mathbb{P}[\tau = x+1 \,|\, \tau > x] = 1 - a_{x+2}/a_{x+1} = p_{x+1}/a_{x+1}$.

(b) The expected return time from state 0 to state 0 is $\mathbb{E}[\tau]$, so the Markov chain is positive recurrent if and only if $\mathbb{E}[\tau] < +\infty$.
Let us show that $\mu : \{0, \dots, M-1\} \to \mathbb{R}_+$ defined by $\mu(x) = a_{x+1}$ is a stationary measure. Indeed, let us check that $\mu \underline{\underline{P}} = \mu$. Note that there is an equation corresponding to each column of $\underline{\underline{P}}$, namely $\mu(x) = \sum_{y=0}^{M-1} \mu(y) p(y, x)$ is the equation corresponding to column x, where $x \in \{0, \dots, M-1\}$. However, these M equations are redundant, because if we sum these M equations, we obtain the trivial $\sum_{x=0}^{M-1} \mu(x) = \sum_{x=0}^{M-1} \mu(x)$, since the matrix $\underline{\underline{P}}$ is stochastic. So it is enough to check that $\mu(x) = \sum_{y=0}^{M-1} \mu(y) p(y, x)$ holds for $x \in \{1, \dots, M-1\}$ and then $\mu(0) = \sum_{y=0}^{M-1} \mu(y) p(y, 0)$ (the equation corresponding to the leftmost column) will automatically follow. Now if $x \in \{1, \dots, M-1\}$, then $\sum_{y=0}^{M-1} \mu(y) p(y, x) = a_x \frac{a_{x+1}}{a_x} = a_{x+1} = \mu(x)$. In order to find the stationary distribution, we need to find the total mass of the stationary measure μ:

$$\sum_{x=0}^{M-1} \mu(x) = \sum_{y=1}^{M} a_y = \sum_{y=1}^{M} \sum_{x=y}^{M} p_x = \sum_{x=1}^{M} \sum_{y=1}^{x} p_x = \sum_{x=1}^{M} x p_x = \mathbb{E}[\tau].$$

Therefore the stationary measure of the Markov chain (α_t) is π, where

$$\pi_x = \frac{\mu(x)}{\mathbb{E}[\tau]} = \frac{\mathbb{P}[\tau > x]}{\mathbb{E}[\tau]}.$$

(c) In this case a renewal time is a recycling time, α_t is the number of wine bottles on the shelf (after possible recycling) in the evening of day t, thus τ is the length of the time interval between two recycling events and

$$p_1 = \frac{1}{2},\ p_2 = \frac{1}{4},\ p_3 = \frac{1}{8},\ p_4 = \frac{1}{16},\ p_5 = \frac{1}{16}, \qquad a_1 = 1,\ a_2 = \frac{1}{2},\ a_3 = \frac{1}{4},\ a_4 = \frac{1}{8},\ a_5 = \frac{1}{16}.$$

Therefore $\mathbb{E}[\tau] = a_1 + \cdots + a_5 = 2 - \frac{1}{16} = \frac{31}{16}$ and

$$(\pi_0, \pi_1, \pi_2, \pi_3, \pi_4) = (\frac{a_1}{\mathbb{E}[\tau]}, \frac{a_2}{\mathbb{E}[\tau]}, \frac{a_3}{\mathbb{E}[\tau]}, \frac{a_4}{\mathbb{E}[\tau]}, \frac{a_5}{\mathbb{E}[\tau]}) = (\frac{16}{31}, \frac{8}{31}, \frac{4}{31}, \frac{2}{31}, \frac{1}{31}).$$

Remark:

So $T(n) = \tau_1 + \cdots + \tau_n$ is the time of the n'th renewal and $N_t = \max\{\, n \,:\, T(n) \le t \,\}$ is the number of renewals up to time t, where $t = 0, 1, 2, \ldots$ We can write

$$\alpha_t = t - T(N_t), \qquad \beta_t = T(N_t + 1) - t, \qquad \gamma_t = \alpha_t + \beta_t = T(N_t + 1) - T(N_t) = \tau_{N_t+1}$$

In words: α_t is the time that has elapsed since the last renewal at time t. β_t is the time until the next renewal at time t. γ_t is the length of the renewal interval that contains t.

Here is a summary of some of the results of HW5.3 and HW6.1, combined with the law of large numbers for Markov chains:

$$\lim_{t\to\infty} \frac{1}{t} \sum_{s=0}^{t-1} \mathbb{1}[\, \alpha_s = x, \beta_s = y \,] = \frac{p_{x+y}}{\mathbb{E}[\tau]}, \quad \lim_{t\to\infty} \frac{1}{t} \sum_{s=0}^{t-1} \mathbb{1}[\, \gamma_s = z \,] = \frac{z p_z}{\mathbb{E}[\tau]}, \quad \lim_{t\to\infty} \frac{1}{t} \sum_{s=0}^{t-1} \mathbb{1}[\, \alpha_s = x \,] = \frac{\mathbb{P}[\tau > x]}{\mathbb{E}[\tau]}$$

Here is an alternative way of obtaining these results. Observe that

$$\sum_{s=0}^{T(n)-1} \mathbb{1}[\, \alpha_s = x, \beta_s = y \,] = \sum_{k=1}^{n} \mathbb{1}[\, \tau_k = x + y \,] \tag{1}$$

$$\sum_{s=0}^{T(n)-1} \mathbb{1}[\, \gamma_s = z \,] = \sum_{k=1}^{n} z \mathbb{1}[\, \tau_k = z \,] \tag{2}$$

$$\sum_{s=0}^{T(n)-1} \mathbb{1}[\, \alpha_s = x \,] = \sum_{k=1}^{n} \mathbb{1}[\, \tau_k > x \,]. \tag{3}$$

(1): in each renewal interval of length $x + y$ there is exactly one time s for which $\alpha_s = x$ and $\beta_s = y$.

(2): these are two ways of counting the total length of renewal intervals of length z.

(3): in every renewal interval of length greater than x there is exactly one time s for which $\alpha_s = x$.

Now by the classical law of large numbers for i.i.d. random variables, we have

$$\lim_{n\to\infty} \frac{1}{T(n)} \sum_{s=0}^{T(n)-1} \mathbb{1}[\, \alpha_s = x, \beta_s = y \,] \overset{(1)}{=} \lim_{n\to\infty} \frac{n}{T(n)} \frac{\sum_{k=1}^{n} \mathbb{1}[\, \tau_k = x + y \,]}{n} = \frac{1}{\mathbb{E}[\tau]} p_{x+y} \tag{4}$$

$$\lim_{n\to\infty} \frac{1}{T(n)} \sum_{s=0}^{T(n)-1} \mathbb{1}[\, \gamma_s = z \,] \overset{(2)}{=} \lim_{n\to\infty} \frac{n}{T(n)} \frac{\sum_{k=1}^{n} z \mathbb{1}[\, \tau_k = z \,]}{n} = \frac{1}{\mathbb{E}[\tau]} z p_z \tag{5}$$

$$\lim_{n\to\infty} \frac{1}{T(n)} \sum_{s=0}^{T(n)-1} \mathbb{1}[\, \alpha_s = x \,] \overset{(3)}{=} \lim_{n\to\infty} \frac{n}{T(n)} \frac{\sum_{k=1}^{n} \mathbb{1}[\, \tau_k > x \,]}{n} = \frac{1}{\mathbb{E}[\tau]} \mathbb{P}[\tau > x] \tag{6}$$

So the *waiting time paradox* is the following phenomenon: if we assume aperiodicity, then convergence to stationarity holds, so we have

$$\lim_{t\to\infty} \mathbb{P}[\tau_{N_t+1} = z] = \lim_{t\to\infty} \mathbb{P}[\gamma_t = z] = \frac{z p_z}{\mathbb{E}[\tau]},$$

in contrast to the trivial $\lim_{n\to\infty} \mathbb{P}[\tau_n = z] = p_z$. The moral of the story is that if $t \gg 1$ then γ_t is essentially a *size-biased sample* from the renewal interval length distribution: t falls into longer intervals with bigger probability.

2. A hunter is trying to shoot a rabbit, the rabbit tries to escape. The rabbit is running around on the locations indexed by $S = \{-10, -9, -8, \dots, 8, 9, 10\}$. The movement of the rabbit is unpredictable, she performs simple, symmetric random walk on S. In each time-step, the rabbit moves to a location adjacent to its current location, and in each time-step, the hunter shoots at the rabbit (first the hunter shoots, then the rabbit moves). Each shot of the hunter is successful with probability $1/100$, but he misses the rabbit with probability $99/100$. As soon as the rabbit reaches the rabbit-holes located at site -10 and site 10, she is safe. If the rabbit initially starts from site 0, what is the probability that she survives this encounter with the hunter?

   Solution:

   Let us add a cemetery state Δ to the state space S, i.e., let $S' = S \cup \{\Delta\}$. So if the rabbit is still alive and she is at location $x \in \{-9, -8 \dots, 8, 9\}$, then she will move to state Δ with probability 0.01, she will move to state $x-1$ with probability 0.495 and she will move to state $x+1$ with probability 0.495.

   (Let us assume that the states -10, 10 and Δ are absorbing states.)

   Denote by $h(x)$ the probability that she survives if she starts from site $x \in S'$. Ultimately we want to find $h(0)$.

   Of course we have $h(-10) = h(10) = 1$ and $h(\Delta) = 0$. We also have

$$h(x) = ah(x-1) + ah(x+1), \qquad x \in \{-9, -8 \dots, 8, 9\}, \qquad a = 0.495 = \frac{99}{200}. \tag{7}$$

   We first try to find the basic solutions of this system of homogeneous linear difference equations:

   We first look for a solution of (7) of form $h(x) = \lambda^x$. Now $\lambda^x = a\lambda^{x-1} + a\lambda^{x+1}$ holds if and only if $1 = a\frac{1}{\lambda} + a\lambda$, or, equivalently, $\lambda = a + a\lambda^2$, or, equivalently $a\lambda^2 - \lambda + a = 0$. Solving this with the quadratic formula we obtain

$$\lambda_{1,2} = \frac{1 \pm \sqrt{1-4a^2}}{2a}.$$

   Note that $\lambda_1 \cdot \lambda_2 = 1$ and if $a = 0.495$ then $\lambda_1 \approx 1.1526$ and $\lambda_2 \approx 0.8676$.

   Thus the general solution of (7) is of form $h(x) = \alpha_1\lambda_1^x + \alpha_2\lambda_2^x$, that is $h(x) = \alpha_1\lambda_1^x + \alpha_2\lambda_1^{-x}$. In order to satisfy the boundary conditions $h(-10) = h(10) = 1$, and also taking into account the built-in symmetries of the exercise, we have to find $\alpha_1 = \alpha_2 = \alpha$ so that $\alpha \cdot \left(\lambda_1^{10} + \lambda_1^{-10}\right) = 1$, this gives $\alpha \approx 0.2283$, so

$$h(x) = \alpha \cdot \left(\lambda_1^x + \lambda_1^{-x}\right), \qquad \alpha \approx 0.2283, \quad \lambda_1 \approx 1.1526.$$

   In particular, $h(0) = \alpha \cdot (1+1) \approx 0.4566$, this is the probability of the event that the rabbit survives.

3. Let $X_1, X_2, \dots$ denote i.i.d. non-negative integer-valued random variables which have the same distribution as the random variable X.

Let $p_k = \mathbb{P}[X = k], k = 0, 1, 2, \dots$. Let us assume that $p_0 > 0$ and that $\sum_{k=0}^{\infty} k p_k = \mathbb{E}(X) > 1$.

Let $Y_0 = 0$ and let us recursively define

$$Y_n = \max\{\, 0, Y_{n-1} + 1 - X_n \,\}, \qquad n = 1, 2, 3, \dots$$

(a) Write down the transition matrix of the irreducible Markov chain (Y_n).

(b) Let $G(z) := \sum_{k=0}^{\infty} z^k p_k$. Show that the equation $z = G(z)$ has exactly one solution $z^* \in (0, 1)$. *Hint:* $G(0) =?$, $G(1) =?$, $G'(1) =?$, what can you say about $G''(z)$ if $0 \le z \le 1$?

(c) Find the stationary distribution of the Markov chain (Y_n).
Hint: We have already found the stationary distribution in the special case when

$$\mathbb{P}[X = 2] = q, \qquad \mathbb{P}[X = 0] = p, \qquad \mathbb{P}[X = 1] = 1 - p - q,$$

see page 81-82 of the scanned lecture notes. The stationary distribution in the general case will look very similar to the stationary distribution in the special case, but note that the method of proof has to be different, because in the general case (Y_n) is not necessarily a birth-and-death chain!

Solution:

(a) Let us define $a_k = \sum_{\ell=k}^{\infty} p_\ell$. The transition matrix on the state space $\{0, 1, 2, 3, \dots\}$ is

$$\underline{\underline{P}} = \begin{pmatrix} a_1 & p_0 & 0 & 0 & 0 & \dots \\ a_2 & p_1 & p_0 & 0 & 0 & \dots \\ a_3 & p_2 & p_1 & p_0 & 0 & \dots \\ a_4 & p_3 & p_2 & p_1 & p_0 & \\ \vdots & \vdots & \vdots & \vdots & & \ddots \end{pmatrix}.$$

(b) We proved this in class when we discussed the extinction probability of a supercritical branching process, c.f. page 103 of the scanned lecture notes.

(c) Let us show that $\mu : \mathbb{N} \to \mathbb{R}_+$ defined by $\mu(x) = z_*^x$ is a stationary measure. Indeed, let us check that $\mu \underline{\underline{P}} = \mu$. Note that there is an equation corresponding to each column of $\underline{\underline{P}}$, namely $\mu(x) = \sum_{y=0}^{\infty} \mu(y) p(y, x)$ is the equation corresponding to column x, where $x \in \{0, 1, 2, \dots\}$. However, these equations are redundant, because if we sum them, we obtain the trivial $\sum_{x=0}^{\infty} \mu(x) = \sum_{x=0}^{\infty} \mu(x)$, since the infinite matrix $\underline{\underline{P}}$ is stochastic. So it is enough to check that $\mu(x) = \sum_{y=0}^{\infty} \mu(y) p(y, x)$ holds for $x \ge 1$ and then $\mu(0) = \sum_{y=0}^{\infty} \mu(y) p(y, 0)$ (the equation corresponding to the leftmost column) will automatically follow.
Now if $x \in \{1, 2, 3, \dots\}$, then

$$\sum_{y=0}^{\infty} \mu(y) p(y, x) = \mu(x-1) p_0 + \mu(x) p_1 + \mu(x+1) p_2 + \cdots = z_*^{x-1} G(z_*) = z_*^{x-1} z_* = z_*^x = \mu(x).$$

In order to find the stationary distribution π, we need to normalize the measure μ by its total mass. We obtain

$$\pi(x) = (1 - z_*) z_*^x, \qquad x = 0, 1, 2, \dots,$$

i.e., the stationary distribution is $\mathrm{GEO}(1 - z_*)$.

Remark: Returning to the special case mentioned in the hint of part (c), if $\mathbb{P}[X = 2] = q$, $\mathbb{P}[X = 0] = p$ and $\mathbb{P}[X = 1] = 1 - p - q$, then $\mathbb{E}[X] = 1 - p + q$, thus $\mathbb{E}[X] > 1$ if and only if $q > p$. Also note that in this case we have

$$G(z) = p + (1 - p - q) z + q z^2,$$

so if we want to find z^* then we need to solve the quadratic equation $z = p + (1 - p - q) z + q z^2$, which is equivalent to $q z^2 + (-p - q) z + p = 0$. The solutions of this equation are $z_1 = 1$ and $z_2 = p/q \in (0, 1)$, thus $z^* = p/q$ and thus the stationary distribution is $\mathrm{GEO}(1 - p/q)$, in agreement with page 81-82 of the scanned lecture notes.