Stoch. Proc. HW assignment 7. Due Friday, October 27 at start of class

Note: Each of the 3 questions is worth 10 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises with a horizontal line. Highlight the final answer. Paper format is preferred, but if you submit your homework electronically, pdf format is preferred.

1. Consider a graph G with vertex set $\mathbb{Z}^d$, where a pair of vertices $\underline{x} = (x_1, \dots, x_d)$ and $\underline{y} = (y_1, \dots, y_d)$ are connected by and edge if and only if $\|\underline{x} - \underline{y}\|_1 = d$ and $\|\underline{x} - \underline{y}\|_\infty = 1$, where $\|\underline{v}\|_1 = \sum_{j=1}^d |v_j|$ and $\|\underline{v}\|_\infty = \max_{1 \le j \le d} |v_j|$. Consider a random walk (X_n) on this graph which starts at $X_0 = \underline{0} = (0, \dots, 0)$, i.e., the origin. The goal of this exercise is to decide whether the Markov chain (X_n) is recurrent or transient (the answer will depend on the value of d).

   Let us also consider the stochastic process
   $$Y_n = \underline{\eta}_1 + \dots + \underline{\eta}_n,$$
   where the increments $(\underline{\eta}_i)_{i=1}^\infty$ are i.i.d. $\mathbb{Z}^d$-valued random variables and each increment $\underline{\eta}_i$ has the same distribution as the random vector $\underline{\eta} = (\eta_1, \dots, \eta_d)$, where $\eta_1, \dots, \eta_d$ are i.i.d. with distribution
   $$\mathbb{P}(\eta_j = 1) = \mathbb{P}(\eta_j = -1) = \frac{1}{2}, \qquad j \in \{1, \dots, d\}.$$

   (a) Show that the stochastic processes (X_n) and (Y_n) have the same law.
   Hint: For the definition of *law*, see page 9 of the scanned lecture notes.

   (b) If $d = 1$, give an explicit formula for the return probability $p^{(n)}(0,0) = \mathbb{P}(X_n = 0)$ for any $n \in \mathbb{N}$.
   Hint: Note that (X_n) is a periodic Markov chain.

   (c) Give an explicit formula for the return probability $p^{(n)}(\underline{0}, \underline{0}) = \mathbb{P}(X_n = \underline{0})$ for any $d \in \mathbb{N}_+$.
   Hint: The d-dimensional answer can be easily expressed using the one-dimensional answer.

   (d) Use Stirling's formula to find $\alpha > 0$ for which $p^{(2n)}(0,0) \asymp n^{-\alpha}$ in the one-dimensional case.
   Hint: Stirling's formula: $n! \asymp e^{-n} n^{n+\frac{1}{2}}$, where $a(n) \asymp b(n)$ means that there exist $0 < c \le C < +\infty$ such that for each $n \in \mathbb{N}_+$, we have $c \le \frac{a(n)}{b(n)} \le C$.

   (e) For which values of d is the Markov chain (X_n) recurrent? For which values of d is it transient?
   Hint: Use the characterisation of recurrence stated on page 75 of the scanned lecture notes.

2. Let us consider a *critical* Galton-Watson branching process (X_n) with GEO $\left(\frac{1}{2}\right)$ offspring distribution and $X_0 = 1$. Denote by $G_n(z)$ the generating function of X_n.

   (a) Use induction on n to find a simple explicit formula for G_n for any $n \in \mathbb{N}$.

   (b) Find $\mathbb{E}(X_n)$.

   (c) Find $\mathbb{P}[X_n = k]$ for each $k = 0, 1, 2, \dots$ *Hint:* $G_n(z) = \sum_{k=0}^\infty \mathbb{P}[X_n = k] z^k$.

   (d) Find $\mathbb{P}[X_n = k \mid X_n > 0]$ for each $k = 0, 1, 2, \dots$, i.e., the distribution of the number of individuals in generation n under the condition that the branching process did not become extinct by time n.

   (e) Find $\mathbb{E}[X_n \mid X_n > 0]$.

3. Each year one hundred thousand students graduate from high school. Each high school student tries to become an astronaut with probability 0.0001. The astronaut entrance exam has two rounds: the first round tests the physical aptitude of the candidate, while the second round is an IQ test. Let us assume that these qualities are independent. A candidate passes the first exam with 20% chance and the second exam with 50% chance. Give simple closed formulas for the following probabilities:

   (a) What is the probability that next year at most one person passes the astronaut entrance exam?

   (b) This year 10 people passed the first round. What is the chance that at least two of them gets accepted?

   (c) Last year n high school graduates became astronauts. What is the probability that k candidates failed the entrance exam? (You can assume that n and k are not very big numbers)

   Hint: In order to answer some (but not all) of these questions, you will need to use Poisson approximation.