Stoch. Proc. HW assignment 7. Due Friday, October 27 at start of class

1. Consider a graph G with vertex set $\mathbb{Z}^d$, where a pair of vertices $\underline{x} = (x_1, \ldots, x_d)$ and $\underline{y} = (y_1, \ldots, y_d)$ are connected by and edge if and only if $\|\underline{x} - \underline{y}\|_1 = d$ and $\|\underline{x} - \underline{y}\|_\infty = 1$, where $\|\underline{v}\|_1 = \sum_{j=1}^d |v_j|$ and $\|\underline{v}\|_\infty = \max_{1 \le j \le d} |v_j|$. Consider a random walk (X_n) on this graph which starts at $X_0 = \underline{0} = (0, \ldots, 0)$, i.e., the origin. The goal of this exercise is to decide whether the Markov chain (X_n) is recurrent or transient (the answer will depend on the value of d).

   Let us also consider the stochastic process

$$Y_n = \underline{\eta}_1 + \cdots + \underline{\eta}_n,$$

   where the increments $(\underline{\eta}_i)_{i=1}^\infty$ are i.i.d. $\mathbb{Z}^d$-valued random variables and each increment $\underline{\eta}_i$ has the same distribution as the random vector $\underline{\eta} = (\eta_1, \ldots, \eta_d)$, where $\eta_1, \ldots, \eta_d$ are i.i.d. with distribution

$$\mathbb{P}(\eta_j = 1) = \mathbb{P}(\eta_j = -1) = \frac{1}{2}, \qquad j \in \{1, \ldots, d\}.$$

   (a) Show that the stochastic processes (X_n) and (Y_n) have the same law.

   (b) If $d = 1$, give an explicit formula for the return probability $p^{(n)}(0,0) = \mathbb{P}(X_n = 0)$ for any $n \in \mathbb{N}$.

   (c) Give an explicit formula for the return probability $p^{(n)}(\underline{0}, \underline{0}) = \mathbb{P}(X_n = \underline{0})$ for any $d \in \mathbb{N}_+$.

   (d) Use Stirling's formula to find $\alpha > 0$ for which $p^{(2n)}(0,0) \asymp n^{-\alpha}$ in the one-dimensional case.
   Hint: Stirling's formula: $n! \asymp e^{-n} n^{n+\frac{1}{2}}$, where $a(n) \asymp b(n)$ means that there exist $0 < c \le C < +\infty$ such that for each $n \in \mathbb{N}_+$, we have $c \le \frac{a(n)}{b(n)} \le C$.

   (e) For which values of d is the Markov chain (X_n) recurrent? For which values of d is it transient?

Solution:

(a) Denote by $H = \{\underline{v} \in \mathbb{Z}^d : \|\underline{v}\|_\infty = 1, \|\underline{v}\|_1 = d\}$. If $(v_1, \ldots, v_d) = \underline{v} \in H$ then we must have $|v_1| = |v_2| = \cdots = |v_d| = 1$, thus $v_i \in \{-1, +1\}$ for all $i \in \{1, \ldots, d\}$. Therefore $H = \{-1, +1\}^d$. In particular, we have $|H| = 2^d$ and thus if $\underline{\eta} = (\eta_1, \ldots, \eta_d)$ is uniformly distributed on H then $\eta_1, \ldots, \eta_d$ are i.i.d. with distribution $\mathbb{P}(\eta_j = 1) = \mathbb{P}(\eta_j = -1) = \frac{1}{2}$ for any $j \in \{1, \ldots, d\}$. Therefore, both (X_n) and (Y_n) are Markov processes with $X_0 = Y_0 = \underline{0}$, and for any $\underline{x}, \underline{y} \in \mathbb{Z}^d$ we have

$$\mathbb{P}\Big(X_{n+1} = \underline{y} \,|\, X_n = \underline{x} \Big) = \frac{\mathbb{1}[\, \underline{y} - \underline{x} \in H \,]}{|H|} = \mathbb{P}\Big(Y_{n+1} = \underline{y} \,|\, Y_n = \underline{x} \Big)$$

We see that (X_n) and (Y_n) are both Markov chains with the same state space, transition rule and initial state, hence have the same law.

(b) In the one-dimensional case, the process (X_n) is just a simple symmetric random walk on $\mathbb{Z}$. The walker can only return to the origin in an even number of steps, thus $p^{(2n+1)}(0,0) = 0$. Now we calculate $p^{(2n)}(0,0)$. In order to return to the origin, the walker has to make exactly n steps to the left and exactly n steps to the right. There are $\binom{2n}{n}$ nearest neighbour trajectories of length $2n$ which make exactly n steps to the left and exactly n steps to the right, and the probability of seeing a particular trajectory is $\left(\frac{1}{2}\right)^{2n}$. Therefore $p^{(2n)}(0,0) = \binom{2n}{n} 4^{-n}$.

(c) We only need to realize that each coordinate of the d-dimensional (X_n) is a one-dimensional simple symmetric random walk and that these d one-dimensional random walks are independent. Now the d-dimensional (X_n) returns to $\underline{0}$ in k steps if and only if each coordinate returns to 0 in k steps. Therefore $p^{(2n+1)}(\underline{0}, \underline{0}) = 0$ and $p^{(2n)}(\underline{0}, \underline{0}) = \left(\binom{2n}{n} 4^{-n}\right)^d$

(d) $p^{(2n)}(0,0) = \frac{(2n)!}{(n!)^2} 4^{-n} \asymp \frac{e^{-2n}(2n)^{2n+\frac{1}{2}}}{(e^{-n} n^{n+\frac{1}{2}})^2} 4^{-n} \asymp \frac{n^{2n+\frac{1}{2}}}{(n^{n+\frac{1}{2}})^2} = n^{-1/2}$, thus $\alpha = 1/2$.

(e) Using (c) and (d), in the d-dimensional case we have $p^{(2n)}(\underline{0}, \underline{0}) \asymp n^{-d/2}$. Thus $\sum_{k=0}^\infty p^{(k)}(\underline{0}, \underline{0}) \asymp \sum_{n=1}^\infty n^{-d/2}$. If $d = 1$ or $d = 2$ then $\sum_{n=1}^\infty n^{-d/2} = +\infty$ and thus (X_n) is recurrent, but if $d \ge 3$ then $\sum_{n=1}^\infty n^{-d/2} < +\infty$ and thus (X_n) is transient. In both cases, we used the equivalent characterisation of recurrence stated on page 75 of the scanned lecture notes.

Remark: The result we have just proved is a variant of the famous *Pólya's recurrence theorem.* Anecdotally speaking: a drunk person always finds his way back home (since the $d = 2$ case is recurrent) but a drunken bird might get lost forever (since the $d = 3$ case is transient).

2. Let us consider a *critical* Galton-Watson branching process (X_n) with GEO $\left(\frac{1}{2}\right)$ offspring distribution and $X_0 = 1$. Denote by $G_n(z)$ the generating function of X_n.

   (a) Use induction on n to find a simple explicit formula for G_n for any $n \in \mathbb{N}$.

   (b) Find $\mathbb{E}(X_n)$.

   (c) Find $\mathbb{P}[X_n = k]$ for each $k = 0, 1, 2, \dots$ *Hint:* $G_n(z) = \sum_{k=0}^{\infty} \mathbb{P}[X_n = k] z^k$.

   (d) Find $\mathbb{P}[X_n = k \,|\, X_n > 0]$ for each $k = 0, 1, 2, \dots$, i.e., the distribution of the number of individuals in generation n under the condition that the branching process did not become extinct by time n.

   (e) Find $\mathbb{E}[X_n \,|\, X_n > 0]$.

Solution:

(a) $G_0(z) = z$
$G_1(z) = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{k+1} z^k = \frac{1}{2}\frac{1}{1-z/2} = \frac{1}{2-z}$
$G_2(z) = G_1(G_1(z)) = \frac{1}{2-1/(2-z)} = \frac{2-z}{3-2z}$
$G_3(z) = G_2(G_1(z)) = \frac{2-1/(2-z)}{3-2/(2-z)} = \frac{3-2z}{4-3z}$
There is a pattern here. We will prove $G_n(z) = \frac{n-(n-1)z}{(n+1)-nz}$ by induction:

$$G_{n+1}(z) = G_n(G_1(z)) = \frac{n-(n-1)/(2-z)}{(n+1)-n/(2-z)} = \frac{2n-nz-n+1}{2n+2-(n+1)z-n} = \frac{(n+1)-nz}{(n+2)-(n+1)z}.$$

(b) $G_n'(z) = \frac{1}{(nz-(n+1))^2}$, thus $\mathbb{E}(X_n) = G_n'(1) = 1$.

(c)

$$G_n(z) = G_n(0) + (G_n(z) - G_n(0)) = \frac{n}{n+1} + \frac{1}{(n+1)^2}\frac{z}{1-\frac{n}{n+1}z} =$$
$$\frac{n}{n+1} + \frac{z}{(n+1)^2}\sum_{\ell=0}^{\infty}\left(\frac{n}{n+1}z\right)^{\ell} =$$
$$\frac{n}{n+1} + \frac{1}{(n+1)^2}\sum_{\ell=0}^{\infty}\left(\frac{n}{n+1}\right)^{\ell} z^{\ell+1} = \frac{n}{n+1} + \sum_{k=1}^{\infty}\frac{n^{k-1}}{(n+1)^{k+1}}z^k,$$

therefore $\mathbb{P}[X_n = 0] = \frac{n}{n+1}$ and $\mathbb{P}[X_n = k] = \frac{n^{k-1}}{(n+1)^{k+1}}$ if $k = 1, 2, \dots$

(d) Denote by $p_{n,k} = \mathbb{P}[X_n = k \,|\, X_n > 0]$. First note that $\mathbb{P}[X_n > 0] = 1 - \mathbb{P}[X_n = 0] = \frac{1}{n+1}$.
We have $p_{n,0} = 0$ and $p_{n,k} = \frac{\mathbb{P}[X_n=k]}{\mathbb{P}[X_n>0]} = \frac{n^{k-1}}{(n+1)^k} = \frac{1}{n+1}\left(1-\frac{1}{n+1}\right)^{k-1}$ if $k = 1, 2, 3, \dots$
In other words, the distribution of the number of individuals in generation n under the condition that the branching process did not become extinct by time n is (optimistic) GEO $\left(\frac{1}{n+1}\right)$.

(e) $\mathbb{E}[\,X_n \,|\, X_n > 0\,] = n + 1$, since the expected value of optimistic GEO (p) is $1/p$.

Remark: If $p \neq \frac{1}{2}$ then it is not so easy to guess the form of the generating function $G_n(z)$ of X_n. First note that if $f(z) = \frac{az+b}{cz+d}$, then we can associate the matrix $\underline{\underline{A}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with f, and it is straightforward to check that if $\underline{\underline{B}}$ is the matrix associated to another such function g, then the matrix associated to $f \circ g$ will be $\underline{\underline{A}}\,\underline{\underline{B}}$. Now the matrix associated to $G(z)$ is $\underline{\underline{C}} = \begin{pmatrix} 0 & p \\ p-1 & 1 \end{pmatrix}$, thus by diagonalizing $\underline{\underline{C}}$ we obtain a formula for $G_n(z)$, i.e., the function associated to $\underline{\underline{C}}^n$:

$$G_n(z) = \frac{\Big(p(1-p)^n - (1-p)p^n\Big)z + \Big(p^{n+1} - p(1-p)^n\Big)}{\Big((1-p)^{n+1} - (1-p)p^n\Big)z + \Big(p^{n+1} - (1-p)^{n+1}\Big)}$$

Let us denote $\gamma = \frac{1-p}{p}$. We have $\mathbb{P}[X_n = 0] = \frac{1-\gamma^n}{1-\gamma^{n+1}}$. Now if $p > \frac{1}{2}$ then $\gamma < 1$, the branching process is subcritical and $\mathbb{P}[X_n = 0]$ converges to 1 at an exponential speed as $n \to \infty$ (i.e., the branching process dies out quickly). But if $p < \frac{1}{2}$ then $\gamma > 1$, the branching process is supercritical and $\mathbb{P}[X_n = 0]$ converges to $1/\gamma$, in agreement with page 104-105 of the scanned lecture notes.

If $p \neq \frac{1}{2}$ then the conditional distribution of X_n under the condition that $X_n > 0$ is optimistic $\mathrm{GEO}(p_n^*)$ with $p_n^* = \frac{1-\gamma}{1-\gamma^{n+1}}$. If $p > \frac{1}{2}$ then $\gamma < 1$, the branching process is subcritical and $\lim_{n\to\infty} p_n^* = 1 - \gamma$, thus $\mathbb{E}[\, X_n \,|\, X_n > 0\,] \approx \frac{1}{1-\gamma}$, while if $p < \frac{1}{2}$ then $\gamma > 1$, the branching process is supercritical and $p_n^* \approx \gamma^{-n}$, thus $\mathbb{E}[\, X_n \,|\, X_n > 0\,] \approx \gamma^n$, this is a huge number: in the supercritical case the population grows exponentially (if it does not die out).

In conclusion:

Subcritical case: $\mathbb{P}[X_n > 0] \to 0$ at exponential speed and $\mathbb{E}[\, X_n \,|\, X_n > 0\,] \to C < \infty$ as $n \to \infty$.

Supercritical case: $\mathbb{P}[X_n > 0] \to c > 0$ and $\mathbb{E}[\, X_n \,|\, X_n > 0\,] \to \infty$ at exponential speed as $n \to \infty$.

Critical case: $\mathbb{P}[X_n > 0] \to 0$ slowly but $\mathbb{E}[\, X_n \,|\, X_n > 0\,] \to \infty$ slowly as $n \to \infty$.

3. Each year one hundred thousand students graduate from high school. Each high school student tries to become an astronaut with probability 0.0001. The astronaut entrance exam has two rounds: the first round tests the physical aptitude of the candidate, while the second round is an IQ test. Let us assume that these qualities are independent. A candidate passes the first exam with 20% chance and the second exam with 50% chance. Give simple closed formulas for the following probabilities:

   (a) What is the probability that next year at most one person passes the astronaut entrance exam?

   (b) This year 10 people passed the first round. What is the chance that at least two of them gets accepted?

   (c) Last year n high school graduates became astronauts. What is the probability that k candidates failed the entrance exam? (You can assume that n and k are not very big numbers)

Solution: Denote by X the number of aspiring astronauts among high school students in a year.

$X \sim \text{BIN}(n, p)$ with $n = 10^5$ and $p = 10^{-4}$, thus $\mathbb{E}(X) = np = 10$.

Clearly $n \gg 1$ and $p \ll 1$ and $\lambda := np \asymp 1$, thus it makes sense to use Poisson approximation and assume that $X \sim \text{POI}(10)$.

(a) A candidate passes the first exam with $\frac{1}{5}$ chance and the second exam with $\frac{1}{2}$ chance, and we assumed independence, so the probability that a candidate passes both exams is $\frac{1}{10}$. If we color each candidate with colors „pass" and „fail" and if we denote by X_2 the number of candidates with „pass" color, then by the coloring property of Poisson distribution we have $X_2 \sim \text{POI}(10 \cdot \frac{1}{10})$, thus the probability that next year at most one person passes the astronaut entrance exam is

$$\mathbb{P}(X_2 \leq 1) = \mathbb{P}(X_2 = 0) + \mathbb{P}(X_2 = 1) = e^{-1}\frac{1^0}{0!} + e^{-1}\frac{1^1}{1!} = 2e^{-1}.$$

(b) If 10 people passed the first round, then the number of people Y passing the second round has binomial distribution: $Y \sim \text{BIN}(10, \frac{1}{2})$. It makes no sense to use Poisson approximation, since the number of trials is equal to ten (not too big) and the chance of a successful trial is equal to a half (not too small). Thus the chance that at least two out of 10 gets accepted is

$$\mathbb{P}(Y \geq 2) = 1 - \mathbb{P}(Y < 2) = 1 - \mathbb{P}(Y = 0) - \mathbb{P}(Y = 1) =$$
$$1 - \binom{10}{0}\left(\frac{1}{2}\right)^0\left(1 - \frac{1}{2}\right)^{10} - \binom{10}{1}\left(\frac{1}{2}\right)^1\left(1 - \frac{1}{2}\right)^9 = 1 - \frac{11}{1024}.$$

(c) Recall that we denote by X the number of aspiring astronauts among high school students in a year and that $X \sim \text{POI}(10)$. If we color each candidate with colors „pass" and „fail" and if we denote by X_1 the number of candidates with „fail" color and X_2 the number of candidates with „pass" color, then by the coloring property of Poisson distribution we have $X_1 \sim \text{POI}(10 \cdot \frac{9}{10})$ and $X_2 \sim \text{POI}(10 \cdot \frac{1}{10})$, moreover X_1 and X_2 are *independent*, thus conditioning on the occurrence of the event $\{X_2 = n\}$ has no effect on the probability of the event $\{X_1 = k\}$. Thus

$$\mathbb{P}(X_1 = k \,|\, X_2 = n) = \mathbb{P}(X_1 = k) = e^{-9}\frac{9^k}{k!}.$$