Stoch. Proc. HW assignment 8. Due Friday, November 3 at start of class

Note: Each of the 3 questions is worth 10 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises with a horizontal line. Highlight the final answer. Paper format is preferred, but if you submit your homework electronically, pdf format is preferred.

1. Consider a time-homogeneous Poisson point process with rate λ on $\mathbb{R}_+$. $T(n)$ denotes the time of the n'th arrival, N_t denotes the number of arrivals up to time t and $N_{(s,t]}$ denotes $N_t - N_s$, i.e., the number of arrivals between s and t, where $0 \le s \le t$.

   (a) Find $\mathbb{P}[\, N_4 = k \,|\, N_3 = n \,]$ for any $k, n \in \mathbb{N}$ and $\mathbb{E}[\, N_4 \,|\, N_3 = n \,]$, $\mathrm{Var}[\, N_4 \,|\, N_3 = n \,]$ for any $n \in \mathbb{N}$.

   (b) Find $\mathbb{P}[\, N_{(3,4]} = k \,|\, N_5 = n \,]$ for any $k, n \in \mathbb{N}$ and $\mathbb{E}[\, N_{(3,4]} \,|\, N_5 = n \,]$, $\mathrm{Var}[\, N_{(3,4]} \,|\, N_5 = n \,]$, $n \in \mathbb{N}$.

   (c) Find $\mathbb{P}[\, N_{(3,6]} = k \,|\, N_4 = n \,]$ for any $k, n \in \mathbb{N}$ and $\mathbb{E}[\, N_{(3,6]} \,|\, N_4 = n \,]$, $\mathrm{Var}[\, N_{(3,6]} \,|\, N_4 = n \,]$, $n \in \mathbb{N}$.

2. The lifetime of a light bulb has $\Gamma[2, 1]$ distribution, in other words the density function of the lifetime of a light bulb is $te^{-t}\mathbb{1}[t \ge 0]$. If a light bulb burns out, I immediately replace it with a new one. At time zero, I start with a new light bulb.

   (a) Find the density function of the time when the third light bulb burns out.

   (b) Find the probability that at time $t = 5$ the third light bulb is on.

   (c) Denote by β_t the remaining lifetime of the lightbulb that is on at time t. In other words, β_t is the length of the time interval that starts with t and ends with the next light bulb-switch. Find $\mathbb{P}(\beta_t \ge s)$ for any $s, t \in \mathbb{R}_+$. Find the density function $f_t(s)$ of the random variable β_t for any $t \in \mathbb{R}_+$. Find $\lim_{t\to\infty} f_t(s)$.

   Hint: The questions (a),(b),(c) become much easier if you find the PPP hidden in the exercise!

3. It rains one hundred times a year on average. Let us assume that the storms are instantaneous and that they arrive according to a PPP. An old gardener waters his garden if it has not been watered (by either rain or himself) in the last 48 hours.

   (a) What is the distribution of the number of storms between two manual waterings?

   (b) What is the expected time that elapses between two manual waterings?

   (c) Roughly how many times does he have to water his garden manually this year?